
SPEED AND FUN WITH FIGURES

T. O'CONOR SLOANE, PH.D., LL.D.

J. E. THOMPSON, B.S. IN E.E., A.M.

H. E. LICKS



NEW YORK

D. VAN NOSTRAND COMPANY, INC.

250 FOURTH AVENUE

FOREWORD

This new publication in the field of mathematics has been designed to combine in one volume those methods and techniques which will give a greater skill and dexterity in calculations of all kinds. To this end, three fundamentally important approaches have been united in this book.

The first approach consists of methods for performing arithmetical calculations quickly, including ways of solving problems mentally. The next logical step in making calculations quickly is by means of a mechanical device—the slide rule. This is fully treated in the second part of this book. Finally, the most effective way of all to achieve facility and skill in the use of figures is by the many recreations and pastimes which have been developed as a source of never-ending enjoyment and pleasure for those interested in mathematics. This approach constitutes the final section of this work which, therefore, contains three effective and yet entirely different means of gaining true mathematical skill.

THE PUBLISHERS

February 1st, 1939

RAPID ARITHMETIC

QUICK AND SPECIAL METHODS IN ARITHMETICAL CALCULATION
TOGETHER WITH A COLLECTION OF PUZZLES AND CURIOSITIES
OF NUMBERS

BY

T. O'CONOR SLOANE, PH.D., LL.D.

Author of "Arithmetic of Electricity," Standard Electrical Dictionary,"
"Elementary Electrical Calculations," and
numerous scientific papers

Copyright, 1922
By D. VAN NOSTRAND COMPANY

All Rights Reserved

*This book or any part thereof may not
be reproduced in any form without
written permission from the publishers.*

PRINTED IN THE U. S.

PREFACE.

There are many things in arithmetic which receive little or but scant treatment in the ordinary text books. If one method of doing an operation is given, it is considered enough. But it is certainly interesting to know that there are a dozen or more methods of adding, that there are a number of ways of applying the other three primary rules, and to find that it is quite within the reach of anyone to add up two columns simultaneously. The multiplication table for some reason stops abruptly at twelve times; it is not hard to carry it on to or at least towards twenty times. Taking up the question of exponents, it is not going too far to assert that many college graduates do not understand the meaning of a fractional exponent, and as few can tell why any number great or small raised to the zero power is equal to one, when it seems as if it ought to be equal to zero.

The multiplication table can be made to give the most curious relations between its constituent figures and quantities, and of these only a part are given here, for one can play a regular game of solitaire with the multiplication table.

The book now in the reader's hands has been a wonderfully interesting work in its preparation. The sources of information and the authorities appealed to were many and in a number of cases were little known. The collection of the matter presented here was a sort of gleaning, picking

up what others had left. It was also a selective task, the accumulating of the best from many sources.

A reference to the table of contents will show that this preface tells only of a small part of what is to be found in the book. In a certain sense this work is a supplement to the ordinary text book of arithmetic. But it is more than that; it will be found of practical application in real work, for by applying the methods to be found in its pages a greater command of arithmetical operations will be acquired and quick ways of calculating will result.

Much that is amusing in the way of oddities and recreations in the science of numbers will be met with in its pages.

The compiler hopes that his mixture of the useful with the lighter phases of his subject will prove acceptable to the reader.

CHAPTER I.

NOTATION AND SIGNS.

Arabic Notation. The Decimal Point. The Number One. Signs in Arithmetic. Decimal Fractions. The Arithmetical Complement. Placing Symbols and Numbers1-10

CHAPTER II.

ADDITION.

Notes on Addition and its Theory. The Addition Table. Carrying One. How to Add. Different Ways of Adding. Bookkeeper's Addition. Group Addition. Index Addition. Period Addition. Addition by Combination. Addition by Average Multiplication. Addition by Multiplication. Decimal Addition. Two and Three Column Addition. Left Hand Addition. Addition without Carrying from Column to Column. Decimalized Addition. Sight Reading in Addition. Inverted or Left Hand Addition. Complementary Addition11-31

CHAPTER III.

SUBTRACTION.

Subtraction. Principles of Subtraction. Decimalized Subtraction. Subtraction in Couples. Subtraction by Inspection. Subtraction by Addition. Inverted or Left Hand Subtraction. Complement Subtraction. Subtraction of Sums. Complement Subtraction of Sums. A Property of Subtraction32-45

CHAPTER IV.

MULTIPLICATION.

Multiplication a Short Method of Addition. The Multiplication Table. Extending the Multiplication Table. Multiplication by Double or Two Digit Numbers. Multiplication of Two Digit Numbers. Multiplication of Three Digit and Higher Numbers. Increment Multiplication. Another Method of Increment Multiplication. Special Method of Multiplying Three Digit Numbers. Multiplication of Special Cases of Polynomials. Inverted or Left Hand Multiplication. Factorial or Proportional Multiplication. Aliquot Part Multiplication. Practical Use of Aliquot Parts in Multiplication. Factor Multiplying. To Multiply by Nine. To Multiply by Eleven. To Multiply by One Hundred and Eleven. Complement Multiplication. Multiplication of Numbers Ending with Five. Multiplying by Two Numbers at Once. Multiplying by any Number between Twelve and Twenty. "Multiplying by the 'Teens." Cross Multiplication. Slide Multiplication. Excess of Nines Proof of Multiplication. Curiosities of the Multiplication Table. A Curious Method of Multiplying. Oddities in Multiplication. Finger Multiplication46-80

CHAPTER V.

DIVISION.

Factor Division. Abbreviated Long Division. Italian Method of Long Division. Excess of Nine in Division. Remainders in Division. Divisibility of Numbers. Special Cases of Division. Dividing by Ninety-Nine. Lewis Carroll's Short Cut81-93 .

CHAPTER VI.

FRACTIONS.

Vulgar Fractions. Meaning of the Fractional Bar. Changing the Value of a Fraction. Reducing Fractions to the Same Class or to a Common Denominator. Addition and Subtraction of Fractions. Multiplication and Division of Fractions. Conversion of Vulgar Fractions into Decimal Fractions94-100

CHAPTER VII.

THE DECIMAL POINT.

Errors Due to Misplaced Decimal Point. Addition of Decimals. Subtraction of Decimals. Multiplication of Decimals. Division of Decimals101-108

CHAPTER VIII.

INTEREST AND DISCOUNT. PERCENTAGE CALCULATIONS.

Expression of Rate of Interest. Calculating Interest. Short Ways of Calculating Interest. Reduction of Interest Periods. One Day's Interest. Divisors for Rates of Interest. Cancellation in Interest Calculations. Percentage Calculations. Approximate Calculations by Percentages109-117

CHAPTER IX.

POWERS OF NUMBERS.

Powers and Roots. Powers and Roots of Decimal and of Mixed Numbers. Relations of Numbers and Square Roots of their Powers. Terminal Digits of Squares and Numbers. A Curious Fraction. Circular Numbers. Properties of Squares. Property of the Square of Two. Fermat's Last Theorem. Properties of Cubes. Orders of Differences of Powers. Powers in Progressions. Relations of Squares of Two Numbers. The Progression of Cubes. A Curious Property of Two Squares. The Sum of Two Squares. Approximations to Isocles Right Angle Triangles. Short

Way of Raising to High Powers. Extraction of Roots of High Powers. Short Method of Calculating Squares of Large Numbers. Various Ways of Squaring Numbers. McGiffert's Methods of Squaring Numbers. Nexen's Method of Extracting Higher Roots	118-147
---	---------

CHAPTER X.

EXPONENTS.

Exponents of Powers. Fractional Exponents. Zero as an Exponent. Prime Exponents. Negative Exponents. Powers of Ten	148-155
--	---------

CHAPTER XI.

SQUARING THE CIRCLE.

Squaring the Circle. Approximations of Former Times. Metius' Value of π . Shaw's Value. Geometrical Approximations. Aids to Memorizing the Value of π . Curious Determination of π by Probabilities. Squaring the Circle Literally	156-160
--	---------

CHAPTER XII.

MISCELLANEOUS.

Prime Numbers. Properties of Prime Numbers. How to Find Prime Numbers. Amicable Numbers. Law of the Square and the Cube. The Four O'Clock Symbol. The Number 108 in Our Solar and Lunar Systems. Automobile Tires. The Two Clerks. Wine and Water Paradox. Paradox of Squares of Fractions of a Number. Divining the Number Thought of. A Paradox of the Time Card. Remembering Telephone Numbers. Making the Nine Digits in Proper Order Add up to One Hundred. Curious Multiplication. A Unique Number. Curious Multiplication and Addition. Multiplication of Nines. Properties of Numbers. Properties of Nine. Bookkeeper's Error. A Mystery in Money. Divining a Sum of Digits. Other Mystifications. Dividing Multiples of Ten by Eleven	161-184
--	---------

CHAPTER I

NOTATION AND SIGNS

INTRODUCTORY

Arabic Notation

The distinguishing feature of the Arabic notation, used all over the civilized world, is the fixing of values by position. In any integral number the decimal point is assumed to be placed at the right of the number—the right hand digit of the number is taken as being on the left of the omitted decimal point. The digit placed here expresses the number of units in the quantity, after the tens, hundreds and other higher decimals have been taken out. Then the next number to the left gives the tens, the next the hundreds and so on.

All this is simple and elementary. But suppose there were no position system of fixing values; then we would be as badly off as were the old Romans, with their clumsy system of literal notation. To express the number eight hundred and eighty eight, if there were no place or position system, we should have to write, 8 hundreds, 8 tens and 8, instead of simply, 888.

Now compare the number of characters required to write this number in Roman and in Arabic systems. In Roman notation it runs: DCCCLXXXVIII, twelve letters as against the three numerals expressing the same thing in Arabic notation—888.

The Decimal Point

So much for the left of the decimal point; the places on the left are used to express integral numbers. Places on the right of the decimal point are used to express fractions of the kind called decimal fractions, which are simply fractions having ten or some power of ten, for their denominator. Thus .8 indicates eight tenths, .88 indicates eighty eight hundredths and so for all decimal fractions.

A number placed next to the decimal point on the right indicates the number of tenths to be taken, if two places from the decimal point it indicates hundredths, if three places from the decimal point it indicates thousandths and the same system is carried out to any extent.

If a single figure is written, without any decimal point next it, it expresses the number of units. If a single figure is to express tens, hundreds or other decimal product, ciphers are used to indicate its position, which are placed to its right; if a single figure is to express a decimal fraction, ciphers on its left fix its position with reference to the decimal point.

A number placed next to the decimal point on right indicates tenths; placed next to the decimal point on the left, it indicates units. This is inconsistent and suggests that the proper place for the decimal point would be where units are placed. This of course is not practical; logic has to yield to practicability.

The Number One

As the basis of any system of notation is unity, or the number one, its properties may be given here.

Any number whatever raised to the zero power is equal to 1.

It is the only number all of whose powers are equal to itself. 1^2 , 1^3 and any power of 1 is equal to 1.

The powers exceeding 1, of all numbers greater than 1, are greater than the numbers so raised. $2^2=4$, $3^2=9$; the power is always greater than the original number.

The powers exceeding 1, of all quantities smaller than 1 are smaller than the quantities. $(\frac{1}{2})^2=\frac{1}{4}$, $(\frac{1}{3})^2=\frac{1}{9}$.

The product obtained by multiplying together any numbers, each greater than 1, gives a quantity larger than either of the multipliers; if the quantities are less than 1 their product will be a smaller quantity than either of them.

Thus the number 1 stands at a dividing point and quantities greater than unity or 1, present characteristics differing from those of quantities less in value than 1.

Signs in Arithmetic

Signs are used in arithmetic as a sort of short hand to indicate operations to be performed on the numbers or quantities in the operations.

In arithmetic number and quantity are generally to be taken as synonymous.

The meaning of a sign is expressed in English or in a few cases in Latin; the latter gives a shorter expression.

The sign of addition is a rectangular cross, with its members respectively horizontal and vertical, placed between the two numbers to be added. It is almost always expressed as "plus," the Latin word for "more;" it is perfectly proper to render it, "added to." Where more

than two numbers are to be added, a plus sign is placed between each; $2 + 2$ indicates the addition of 2 to 2; $2 + 2 + 3 + 4$ indicates the addition of 2, 2, 3, and 4; the first addition giving a sum of 4, the second a sum of 11.

The result of a completed addition is the sum of the numbers or quantities added. The operation is not correctly called a sum.

The sign of subtraction is a horizontal bar or dash; it is placed between numbers, the last number to be subtracted from the first one. It is usually expressed as "minus," the Latin word for "less," we may say correctly to express it, "diminished by," but minus is always used. $5-4$ reads 5 minus 4, and means four subtracted from five. It will be observed that in stating such a subtraction the larger number is always placed first as $5-4$, five minus four or $6-3$, six diminished by three. The latter expression is not in use to any extent.

The quantity from which a quantity is subtracted is called the minuend, from the Latin, meaning "to be diminished"; the quantity which is subtracted is called the subtrahend, from the Latin, meaning "to be subtracted"; the result of the operation is called the remainder or difference.

What has been said about the larger quantity being placed first, in the expression of subtraction, refers to arithmetic, not to algebra.

Multiplication is indicated by a diagonal cross, placed between the numbers; 4×5 indicates 4 multiplied by 5. In continued multiplication the sign must be placed between each pair of quantities; $4 \times 5 \times 6$ indicates 4 multiplied by 5 multiplied by 6, whose product is 120.

Sometimes a period is used as the sign of multipli-

cation. It is liable to cause confusion, by being taken for the decimal point, yet it is very frequently employed.

If multiplications are set down in full, the upper quantity is called the multiplicand, from the Latin, meaning "to be multiplied"; the lower quantity is called the multiplier; and when the operation is completed the result is called the product. There is no reason for placing the multiplicand above the multiplier; they can change position and rôles without affecting the result of the calculation.

Division is indicated in several ways. One is by a horizontal bar or to save space a diagonal one. The number above the bar is called the dividend, from the Latin, meaning "to be divided"; the number with which it is to be divided is placed below the bar; it is called the divisor. Thus $\frac{6}{3}$ or what is the same thing, $6 \over 3$ indicates 6 divided by 3. The result of a division is called the quotient, from the Latin, meaning "how often."

Another sign of division is possibly derived from this one; it is a horizontal bar with one point over its center and one below its center. $6 \div 3$ indicates 6 divided by 3.

The colon, :, is a sign of ratio and hence of division, but is not used, simply to indicate division, except in special cases.

The horizontal or oblique bar is not always admitted to be the sign of division; the claim is sometimes made that there is a difference between such expressions as $2 \div 4$ and $\frac{2}{4}$. The latter is taken as indicating a fraction only. But if we take such an expression as a/b , it is hard to see how it can be expressed except as " a divided by b ." In the case of numbers the alternative fractional nomenclature is always available, as "two fourths" in the expression or fraction, $\frac{2}{4}$.

Whatever name is given it, $\frac{2}{4}$ means and indicates the division of 2 by 4.

The indication of the higher power of a number is a small figure placed above and to the right; it is called an exponent; 4^2 means the square of 4, which is 16; 5^3 means the third power or cube of 5, which is 125. The little figures, 2 and 3, are exponents, in these instances, of 4 and of 5 respectively.

The term "square" is an abbreviated way of expressing the second power; the term "cube" is the same for the third power; there are no other abbreviations of power expressions.

The radical sign indicates the root of a number. By itself it indicates the square root; if any other root is to be indicated the exponent is placed over it to the left. $\sqrt{16}$ means the square root of 16 which is 4; $\sqrt[4]{16}$ means the fourth root of 16, which is 2.

When there are several numbers required to express a quantity the combination is called an "expression." $2 + 3$ and $3 + 5$ are expressions.

The sign of equality is a pair of horizontal bars parallel to one another, $=$; they read "equal" or "equals." Thus to express the result of adding two quantities, say 2 and 3, we would write, $2 + 3 = 5$, which reads 2 plus 3 equal 5.

A statement such as the above, affirming the equality of two quantities or expressions, is called an equation and sometimes a formula.

Inequality is indicated by a V-shaped symbol placed on its side; the quantity next the apex is stated to be the smaller. $7 > 2$ means that 7 is greater than 2, or that 2 is less than 7. Both come to the same thing; the sign can face the other way; $2 < 7$ means that 2 is less than 7.

Two colons, placed one after the other, are the central sign of a proportion and are read "as," while in the same proportion the single colon is read "is to"; thus $2:4::4:8$ reads 2 is to 4 as 4 is to 8. Sometimes the equality sign, $=$, is used as the central symbol. If this is done in the above proportion we would have: $2:4=4:8$.

Taking the colon as the sign of division our last proportion would read 2 divided by 4 equal 4 divided by 8, which is perfectly correct, and expresses the relation existing between the members of a proportion. The double colon is never taken as a sign of equality, though it might be.

A parenthesis expresses the fact that the quantities within the parenthesis are to be taken as a group and to be treated as a single or individual quantity. $7 - (2 + 3)$ means that the sum of 2 and 3 is to be subtracted from 7; this gives a remainder of 2. The same numbers without the parenthesis, $7 - 2 + 3$ tell us that 2 is to be subtracted from 7 and that 3 is to be added; the result is 8. The parenthesis brings about a totally different result.

Multiplication and division signs take precedence of addition and subtraction signs; the first two signs unite the numbers between which they are placed, as if they were in a parenthesis. Thus $12 - 10 \div 2$ indicates that 10 is to be divided by 2 and the quotient is to be subtracted from 12; this gives 7, as the result. In like manner $4 + 6 \times 3$ indicates that 3 times 6 are to be added to 4, which gives 22; 6×3 act as if in a parenthesis.

Decimal Fractions

A decimal fraction in the broad sense is a fraction whose denominator is a power of 10; $\frac{1}{10}$, $\frac{2}{100}$ and

$\frac{5}{1000}$ are decimal fractions under this definition. Generally the term is restricted to the writing the same class of fractions on the line, using the decimal point.

To do this the numerator of the fraction is written to the right of the decimal point and the denominator, as such, is omitted and is expressed by the relation of the numerator to the decimal point, and this relation is fixed by the use of ciphers, or sometimes simply by the digits in the numerator. Thus if the digits in the numerator are too few to give the proper position or distance from the decimal point to the numerator, ciphers are placed to the right of the point between it and the numerator.

The number of digits or ciphers less one in the denominator of a decimal vulgar fraction gives the number of places in the decimal fraction. $\frac{1}{10}$ is written .1, because there are two digits in the denominator; $\frac{3}{100}$ is written .03, because there are three digits in the denominator.

The Arithmetical Complement

Complement means anything which fills up or completes. If a number is referred to the multiple of 10 immediately above it, the difference between it and that multiple of 10 is its complement. On this basis 2 would be the complement of 18, because the multiple of 10 next to 18 is 20.

Often the higher number to which it is referred is the next higher power of 10.

On this basis the complement of 18 would be 82, for the next power of 10, above 18, is 100.

The arithmetical complement of a decimal fraction is the difference between it and unity or 1.

Thus the arithmetical complement of .55 is .45. This

complement is constantly employed in trigonometrical calculations. It is most easily obtained by subtracting the right hand digit of the decimal from 10 and the other digits from 9. Suppose we want the arithmetical complement of .4658; we proceed as follows: $10 - 8 = 2$; this is the right hand digit of the complement. Then we go on and subtract from 9; $9 - 5$, $9 - 6$ and $9 - 4$ respectively; the result is .5342. The sum of a decimal and its arithmetical complement is always unity or 1.

Placing Symbols and Numbers

To put a symbol or number in front or to the left of another is to prefix; to put a symbol or number to the right of a number is to annex.

If a period is prefixed to 55 it converts it into a decimal, .55. If a 5 is annexed to 55 it gives 555. It is important to distinguish between annexing and adding.

A number to which a sign is affixed is said to be affected by that sign. Thus if we write -6 , it reads minus 6 and the number, 6, is said to be affected by a minus sign.

The positive sign is never prefixed to a number written alone, yet every positive number may be said to be affected by a positive sign understood.

A number with negative sign prefixed, or affected by a negative sign, is said to be a negative number. A number with no sign prefixed is said to be a positive number.

The use of positive and negative numbers and quantities as such, is generally restricted to algebra.

A number with an exponent is said to be affected by the exponent. Thus the numbers 2 and 5 in the ex-

pressions, 2^3 , 5^4 are said to be affected by the exponents, 3 and 4.

The introduction of the plus sign, $+$, and of the minus sign, $-$, is attributed to a German mathematician, Michael Stifel, b.1480, d.1567; the date of their use by him is given as 1544. To him is also credited the radical sign, $\sqrt{}$.

The sign of equality, $=$, originated with Robert Recorde, an English mathematician, b. (about) 1500, d.1558. He first used it in 1557.

The sign of multiplication, $\times$, originated with William Oughtred, an English mathematician, b.1574, d.1660. It appeared in his work, *Clavis Mathematicae*, 1631.

The sign of division, $\div$, is attributed to an English mathematician, Dr. John Pell, b.1610, d.1685.

Addition was originally expressed by writing out the Latin word, plus, by the Italian word, piu, or by the letter, p. Subtraction was indicated by minus, mene, or m. The radical sign succeeded the capital letter, R, as the sign of a root of a number.

The signs of inequality, $>$ and $<$, appear first in a posthumous book of Thomas Harriot, an English mathematician, a contemporary of Oughtred.

The decimal point was introduced by John Napier, (b.1550, d.1617) the famous Scotch mathematician, in the beginning of the seventeenth century.

The decimal system of numbers was introduced into Europe in the tenth century.

CHAPTER II.

ADDITION

Notes on Addition and its Theory

Everyone is supposed to know the multiplication table; this means the ability to name the product of any two numbers within the range of the usual multiplication table, and to do this at once without stopping to think. In the table in question there are, if we count the "one times" part, 144 different multiplications. Some of these are repetitions inverted, such as 3 times 9 and 9 times 3. It is safe to call the products to be remembered 132 in number, as the inverted ones should count as individual products, even if the quantities are identical.

Addition is the first rule of arithmetical operations taught to the child, for numeration is hardly to be called an operation. Yet of all the four basic things in arithmetic, of addition, subtraction, multiplication and division, addition gives the most trouble, is the most subject to error and is the most used of the four. So true is this that while every bank has one or more adding machines, comparatively few consider it worth while to have a multiplying machine. Owing to the extensive introduction of adding machines there is not the same demand for men who are rapid adders, that there once was. But we all have our own adding to do and few have access to an adding machine. It is satisfactory to know that, by proper methods and by analysis of its combinations, the process of addition can be facilitated

to a greater extent than would seem possible. There are a number of different ways of adding up columns of figures, there are a number of special ways in use by individuals, of helping the work along so that many operators have what may be called their own ways of getting at results.

The Addition Table

In the multiplication table there are 144 different multiplications to be remembered and to be known so perfectly that no time or thought is to be given to them when asked or required. In the corresponding addition table there are only 45 additions to be remembered, yet for some reason it is fair to say that they are not as well known in general as are the products of the multiplication table.

The following things are to be noted about the addition of the nine digits to each other, in twos, that is to say in the addition of one digit to another, for all the nine digits.

The total number of such additions is forty five.

Of these additions twenty give single numbers. Such are $2 + 4 = 6$, $3 + 5 = 8$.

Of these additions twenty five give double numbers. Such are $5 + 6 = 11$, $7 + 9 = 16$.

The largest number given by the addition of two digits is eighteen, $9 + 9 = 18$; the figure on the left is 1, this is the figure to be carried when we add; therefore the figure to be carried in addition is never increased in an addition to any number of a single digit by more than 1.

To illustrate this statement suppose 6, 7, 8 and 9 are to be added together. 6 and 7 are 13; this gives 1 to carry. Then comes 3 and 8 giving 11 to which is added the 1

carried from the first addition, giving 21, so that there is 2 to carry. Next 1 is added to 9 giving, with the 2 to carry, 30 as the sum of the four numbers.

There was first 1 to carry, it could not possibly be more than 1. The next number to be carried, due to the addition this time of three figures was 2, one more than the first figure carried, which was 1. Then when the addition of the four was completed the figure in the tens' place was one more than the last figure carried, it was 3.

This leads to the following conclusion: When a column of single digits is added up the successive left hand figures in the ten places can never exceed each other by more than 1 at a time; successive figures in the ten places may be the same.

In the following additions the separate additions are given at the right hand side of the column of amounts to be added.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
9 50	1 22	9 32	9 40
8 41	2 21	8 23	8 31
9 33	2 19	7 15	7 23
7 24	8 17	8	8 16
8 17	9	—	8
9	—	32	—
—	22		40
50			

Column *a* is made up of such figures as involve the carrying of an additional 1 for each adding; this is the most that can be carried and it will be observed that the successive sums are, as regards their figures in the ten places, successively greater by 1, so that the ten place

How to Add

In the modern teaching of reading the effort is made to teach the pupils to learn to know words without the process of spelling. In adding numbers they should be added without naming; thus in adding the figures in column *a*, page 13, you should say to yourself in quick succession 9, 17, 24, 33, 41, 50; do not say to yourself 9 and 8 are 17 and 7 makes 24 and so on.

It is interesting to use the tables just given to test your quickness at addition; if you cannot give the additions of the tables without hesitation and almost without thinking, fast addition and even accurate addition to say the least will be difficult.

Different Ways of Adding

Addition may be done column by column and one digit at a time; this is probably the most usual way of doing it. It is the most obvious and the simplest but is also the slowest. There are two ways of doing it—from above down or from the bottom of the column upwards. To prove the accuracy of the operation it is well to do it first one way and then the other.

Bookkeeper's Addition

A minor variation which can be used to advantage is to write down the sum of each column separately, one sum under the other and each successive sum set one space to the left; then a subsidiary addition gives the total. This is done here:

9938	Addition of first column	30
7827	“ “ second column	8
4119	“ “ third column	26
6826	“ “ fourth column	26
28710	Total	28710

In the left hand addition it is added up in the regular way; in the right hand addition the way just described is carried out.

Group Addition

The different additions of two digits give only 17 different numbers, so it is a very easy matter to know them. It is not easy to give them without stopping to think. When perfectly known they lead to another way of adding, the first step of group adding.

This method consists in adding two or more numbers at a time to two or more others in the vertical columns. Thus the following addition would be done by at once seeing in the 8 and 7 the number 15 and in the 12 9 and 3 above them the number 12; then 12 and 15 would be added, giving 27. This may or may not involve adding double numbers to each other, but this method is made simple by the fact that the sum of two numbers can never have anything higher than 1 in the ten place and nearly half the time will not have even that.

When several columns are to be added by grouping, there will generally be a number to be carried; this may be added to the first group of the next column, or the whole operation may be treated as is shown on later pages.

If the direct adding of such numbers as 15 and 12 to each other is too difficult, then for rapid work proceed as follows: In the last example add 15 to 10 giving 25; then to this 25 add the missing 2 of the 12, for you have already added its 10, reading it off to yourself in this way: 15, 25, 27. This system makes it as simple as ordinary adding. The process of group adding is easier and better in every way than the direct single column way.

Index Adding

Two similar methods of addition are next given in the examples at the side of the pages, which while simple enough are of some value. Every different way of adding up a column of figures, if it has no other value, is of use as a way of testing the accuracy of the operation or, as it is usually put, of "proving it."

Referring to the left hand column it has been added up as follows: Starting at the bottom the figures have been added until they approach 20 so closely that the addition of the next digit above would give 20 or a sum exceeding 20. At this point the last figure of the sum reached is written to the side of the column. A new addition, one with no reference to the one just done, is started and carried on until 20 is again approached; the terminal digit is written and this is repeated until the top is reached. If the top addition has no ten place figure in it, if it is less than 10, nothing is written at the side. Then all the figures at the side are added up, if there are any top figures above the last or uppermost key-figure these are added in also, and in front of the sum is written a number for the tens equal to the number of key-figures written at the side, increased by anything to be carried from the addition of the key-figures.

In the left hand column the lowest three figures added up give 15. The figure next above them is 7, this is not to be added in because the sum would exceed 20. Therefore 5 is written on the side. Starting anew, we add 7 and 8 giving 15; here we must stop for the next figure is 6 which added in would give 21. So stopping here 5 is written on the side. In this way we go on until the top

is reached. As the last or highest figures of the column give a number in excess of 10 the figure 7 of their sum, 17, is written. Now the key-figures are added up; they give 25. We write down the 5 and carrying the 2 add to it 4, or 1 for each key-number. There are four key-numbers, so the total is for the tens, $2 + 4$, which gives 6, and the sum of the whole is 65.

Suppose now that there had been no eight at the top of the column and that after the last key-number had been written only the 9 had been left; then there would have been only three key-numbers, 5, 5 and 8 making 18; to these would have been added the remaining 9, at what would then have been the top of the column; this would have given 27; then there would have been only 2 to carry and to go into the ten place in addition to the 3 of the key-numbers; we would have a total of 57, which would be the true sum of the column without the topmost figure 8.

Period Addition

The next process is done in a similar way. The addition of the figures is continuous this time right up to the

8 .	top, except that when the number, just as before,
9 .	approaches 20 a dot is placed at the side, and the
7 .	addition goes on but with omission of the tens.
3	Thus after the third figure from the bottom is
2 .	reached we make a dot and go on with 5 as a
6	starting figure. Then we have $5 + 7$; this gives
8	12. The next figure is 8, so we put a dot at the
7 .	side and starting with 2 add it to the figures above
6 .	it, 8, 6, and 2, find that a dot goes here where the
3	total is 18; then this 8 is added to the 3 and 7
6	above it and another dot is put down and for the
—	final figures we have 17 for the next to the last
65	

dot and carrying the 7 the top figure gives $7 + 8$ or 15. Here is the last dot and the unit figure 5 to be written down, while for the tens count the dots. There are six so our tens are 6 and the number is 65.

Just as before, if there is no number exceeding ten in the last addition, the numbers are simply added in to make up the units. As in the first of the two methods, there may be 1 to carry at the top.

Addition by Combination

It is sometimes advised that the combinations of numbers adding up to ten be memorized. Such are 3, 3, 4; 1, 3, 6; 2, 3, 5. The two figure combinations are so well known that it is to be presumed that everyone knows them. Anyone can write out the different combinations and study them. It is also recommended to learn the combinations of more figures, such as the eight combinations of four figures which give 20. There are nine combinations of four figures which give 30. The higher combinations are of little use, for the more numbers involved in a combination the rarer it is. The two and three figure combinations are the most useful.

Every such combination which can be memorized is an addition to one's faculties in doing group adding. Group adding should not be confined to sets of only two numbers. All these ways contribute to group adding. It will often be found that expert adders have their own favorite ways of grouping.

Addition by Average Multiplication

When numbers occur, whose middle figure is the average of the lot, multiplying it by the number of figures will give the sum. Suppose 5, 4 and 3 are to be added; 4

is the average of the three figures; therefore $4 \times 3 = 12$ is the sum. Such combinations as this are to be watched for and made use of.

Addition by Multiplication

The following is another way of adding up a column of single figures. By adding and subtracting the figures are all reduced to the same, and a simple multiplication gives the addition provided the same amounts were added as were subtracted. Otherwise a correction must be made. Some examples follow.

$9 - 1 = 8$	$9 - 2 = 7$
$9 - 1 = 8$	$6 + 1 = 7$
$8 = 8$	$3 + 4 = 7$
$7 + 1 = 8$	$4 + 3 = 7$
$7 + 1 = 8$	$8 - 1 = 7$
40 40	30 5 35
	5
	30

The idea is to substitute a simple multiplication for an addition. In the second case as 8 has been added and 3 subtracted, the net result is an addition of 5, which has to be subtracted from 35 to give the answer.

Decimal Addition

The next method may be termed decimal adding. To all of the quantities add such quantities as will make them multiples of 10 or of 100. From the last of the original quantities subtract the sum of the increments or

amounts added. A simplified addition gives the sum required.

Add 9, 7, and 4.

$$9 + 1 = 10$$

$$7 + 3 = 10$$

$$4 - 4 = 0$$

—

20

Add 97, 89, and 49.

$$97 + 3 = 100$$

$$89 + 11 = 100$$

$$49 - 14 = 35$$

—

235

In the first example the increments are 1 and 3, their sum is 4, and this is subtracted from the last of the original quantities, which is also 4. The sum of the numbers in the right hand column gives the sum of the original numbers. In the second example 14 is the sum of the increments, and is subtracted from 49, the last of the original numbers. The sum of the right hand column gives the answer.

Two and Three Column Addition

There are ninety combinations of double numbers, 10, 11, 12 and so on up to 99. If the sum of any two of these can be given without hesitation at sight, then two columns can be added simultaneously. This implies double speed, and there are many computers who can do this. Some can add three columns simultaneously.

The addition of two columns at once can be done by the following method by anyone. It is a method well worth acquirement.

The bottom or top number of the two columns is added to the tens of the number next it, and then the units of the next number are added; this gives the sum of the two. Then the tens of the third number are added and then

the units of the third number giving the sum of the three. This is continued to the end of the columns.

29 To apply this method to the example here given
 34 proceed as follows: Add 70 to 88; this gives 158.
 71 To this add the 1 of the 71, giving 159. To 159
 88 add 30, giving 189, and to this add the 4 of the
 — 34 giving 193. To 193 add 20, giving 213 and to
 222 this add the 9 of the 29 and the total is 222.

A variation can be used and the units can be added in before the tens; thus $88 + 1 = 89$. This takes care of the unit figure of the 71. Then add 70, which makes 159. $159 + 4 = 163$ and adding the 30 of 34 to this gives 193. Then $193 + 9 = 202$ and 20 added to this gives the total 222 as before.

Three columns in width can be added by an extension of the same method. Thus to add 957 to 875 if we start with the hundreds it goes as follows: $957 + 800 = 1757$; $1757 + 70 = 1827$; $1827 + 5 = 1832$.

The three column method is sometimes said to be only useful within the limits of a limited range of numbers; it is good practice however. The two number adding is quite practical for much longer rows of numbers. An example of three column addition follows.

Add the following:

541	$1042 + 500 = 1542$
237	$1002 + 40 = 1042$
764	$1001 + 1 = 1002$
—	$801 + 200 = 1001$
1542	$771 + 30 = 801$
	$764 + 7 = 771$

As the addition was started at the bottom of the columns the sum is reached at the top.

Left Hand Addition

When a number of columns are to be added the addition may commence with the left hand column at the bottom. When completed the adjacent figure of the next column is annexed, not added, to it and the addition is carried right down the second column. The addition or sum of the two columns is written down at the bottom and the next two columns are treated in the same way. If their sum includes only two digits they are set down to the right of the sum of the first two columns; if they comprise three or more digits they are set down one line below the others and with all except the two right hand digits under the right hand digits of the upper sum. The two are then added if there are only the four columns; if there are more, the same process is carried out. The one thing to be kept in mind is that the sum of the first, or right hand column is not added to the second, but is treated as the tens and hundreds of a quantity to which the second column supplies the units.

1798
9788
8967
7899

281
352

In the example the first column adds up to 25; assume that we are adding from below upwards. Then to 25 we annex the 7, which is the top figure of the second column and go down with the addition, 257, 264, 273, 281. The last is the sum of the first and second columns counting from the left and is written down below the line. The same is done for the next two columns; we get 32 for the next to the last column; we annex 8 giving 328, and adding as before we get 328, 336, 343, 352. As this has more than two digits it is written a line below the other addition, its left hand figure under the right hand figure of the other sum. The two are then added.

28452

An interesting variation on the above method is the following. Take the same example to be added.

We start as before at the lower left hand figure and add up the left hand row, which gives 25. Then as before annexing the 7 at the top of the next row which gives 257 as before we finish the row and get 281. Here the difference comes in. We put down 28 only and annex the 9 at the foot of the next column, which gives 19. Then proceeding as before when we reach the top of the third column from the left, we have 42. To this annex the 8 at the top of the right hand column, making 428, and then going down the column to the foot we get as the final sum of the last operation 452, which is put down directly in place. There is here no subsidiary addition; the whole sum is put down in one row, completing the addition. (See example *a*.)

Had there been more than four columns the numbers 28 would have first been put down; then when the second addition was completed the two left hand numbers 45 would be put down, and the 2 would be carried on and annexed to the next column. (See example *b*.)

$$\begin{array}{r}
 (a) \\
 1798 \\
 9788 \\
 8967 \\
 7899 \\
 \hline
 28 \\
 452 \\
 \hline
 28452
 \end{array}$$

$$\begin{array}{r}
 (b) \\
 17986 \\
 97882 \\
 89675 \\
 78991 \\
 \hline
 28 \\
 45 \\
 34 \\
 \hline
 284534
 \end{array}$$

In practice the additions in the last two examples would be written at once on the same line; they are here placed on different lines to illustrate and explain the method.

Addition without Carrying from Column to Column

A very convenient way of adding several columns is the following. The first column, the right hand one, is added up and the sum is written down, under the first and, if of two figures, with its second figure under the second column. If there are three figures in the sum of the first column then its third figure goes under the third column. Next the third column is added up and its sum is set down with its first figure under the third column, its second figure under the fourth column and so on. Suppose there are four columns to be added. Then returning to the second column it is added up and its sum is put down on the same principle on the line below the sums already set down. Finally the fourth column is added up and treated in the same way. A final addition of the two rows of figures gives the total.

8984
8435
9917
—
2216
2512
—
27336

In the example the first two figures of the upper line, 16, are the sum of the first column— $4 + 5 + 7$; the next pair are the sum of the third column— $9 + 4 + 9$. Going down to the next row 12 is the sum of the second column— $8 + 3 + 1$, and 25 is the sum of the fourth column— $8 + 8 + 9$.

If there are three figures in any sum they are set down as described and the only effect is that one more

row is required. Three figures in any sum can only occur when the column is more than twelve figures high. Suppose that the first column of four columns added up 116, the third, 322, the second, 312 and the fourth 125. They would be put down thus:

116
322
312
125
—

In such a case as this it would be much better to set them down in regular order. The method only applies when practically each sum is less than 100; when each sum contains less than three figures.

Another method of adding is shown in the next example.

382925
399098
976878
—

1547781
21112
—

1758901

Starting at the left the column is added up giving 15, which is put down as shown. The next column is added up, it gives 24; this is put down, the 2 under the 5 and the 4 next to the 5 on the same line. The other columns are treated in the same way; their sums are 17, 17, 18 and 21. All are put down obliquely on two lines and the final addition gives the sum of the whole.

Decimalized Addition

For additions of two small quantities a convenient way is to make a decimal out of one of them by addition or subtraction as the case may be. The other number is treated in the exact reverse way, and the two are added. Suppose 97 is to be added to 86; add 3 to the 97 making it 100; treat the 86 in the reverse way and subtract 3 from it, which gives 83, and the sum is instantly seen to be 183. If we only add the 3 to the first number let the other alone and add, the subtraction of 3 from the sum gives the desired total.

$$\begin{array}{r}
 97 + 3 = 100 \\
 86 - 3 = 83 \\
 \hline
 183
 \end{array}$$

Carrying out the same method to a fuller development, leads to the following method of performing addition. Reduce each number to a decimal by adding or subtracting as you please; add the decimals and add or subtract the increments, adding such as you subtracted and subtracting those you added. Thus to add 341, 896 and 302. Add 59 to the first number, add 4 to the next one and subtract 2 from the last. This gives the result shown; the unchanged figures are first given, then the new ones and then the increments to be added or subtracted.

341	+ 59	400
896	+ 4	900
302	- 2	300
1539	61	1600 - 61 = 1539

As 59 and 4 have been added to decimalize the numbers they have to be subtracted from the sum of the decimals, and as 2 has been subtracted it has to be added. So we say 59 and 4 are 63, to be subtracted and 63 less 2 gives 61, the net sum to be subtracted.

Sight Reading in Addition

To be able instinctively to name the sum of several figures is what is sometimes called sight reading. Take the two following additions: 23 to be added to 45 and 59 to be added to 75. Treat each case as two separate

additions and to make this clear we shall write the figures with dots to separate them in couples.

$\begin{array}{r} 2 \cdot 3 \\ 4 \cdot 5 \\ \hline 6 \cdot 8 \\ 68 \end{array}$	$\begin{array}{r} 5 \cdot 9 \\ 7 \cdot 5 \\ \hline 14 \\ 12 \\ \hline 134 \end{array}$
---	--

The eye is supposed to be fixed between the two columns where the dots are; to immediately see a 6 and an 8, making 68, or a 12 and a 14, making 134. Of course the reader will see that this is not a case of straight addition; it is really adding in the one case 60 to 8, and in the other case adding 120 to 14; but it is supposed to be done instinctively. There is a bit of psychology in adding—the calculator who hesitates is lost—the minute you let yourself think and cease to rely on instinct you will fail in rapid work.

Inverted or Left Hand Addition

When two large numbers are to be added to each other the operation is usually begun at the right hand. It is much better to begin at the left hand. This is called inverted or left hand addition.

Add mentally the left hand figures. If the two next to them add up less than 9, write down the sum of the left hand figures. If the two next add up more than 9, carry one, that is increase the sum of the left hand figures by 1 and write it down. If the two next add up 9, then look at those next to them on their right and see how

they add up. If less than 9, write down the addition; if 9 is the sum of the neighboring figures see if their next door numbers are less than 9. If so there is nothing to carry; if over 9 there is 1 to carry; if the sum is 9 then go one more to the right and see if there the sum is less than 9, is equal to 9 or is more than 9, and proceed accordingly. It is perfectly simple when you do it, though it seems a little complicated in the description. The following example will make it clear. Add the following numbers:

$$\begin{array}{r} 7906872 \\ 8293138 \\ \hline 16200010 \end{array}$$

We add mentally, without writing it down, 7 and 8, which are 15. The next numbers on the right add up more than 9, therefore we carry 1 and write down 16. The next numbers are 9 and 2 making 11; next on the right is a sum of 9, the same on its right, and then a third sum of 9 and then a sum of 7 and 3 which is 10; this means the carrying of 1 right down the line giving the three 0's and one to carry to be added to the sum of 9 and 2, making it 12, so we write down the 2 having already carried the 1. We now go back to the 7 and 3; next to them on their right are 8 and 2 to be added; these give 10, so we have to carry 1 to the sum of 7 and 3 making it 11; we write down 1, having already carried the other 1; we then write down for the sum of 2 and 8 a cipher, 0, because we have already carried the 1.

This is a most complicated example; ordinarily it goes much easier, and with a little practice the addition is written down as if you had it by heart.

It is especially in taking out logarithms that this method is recommended, but it is the best way of adding numbers when only two high. It is not so practical when the column is three or more high.

Complementary Addition

The following is a sort of complementary addition, which is very practical and useful. It is a method of adding two columns at once:

Add the following quantities: 78, 54, 89 and 65.

$78 + 22 = 100$	$54 - 22 = 32$	78
$100 + 32 = 132$	$89 - 68 = 21$	54
$132 + 68 = 200$		89
$200 + 21 = 221$		65
$221 + 65 = 286$		286

Its complement, 22, is added to the first quantity, 78, giving 100. The same complement is subtracted from the next quantity to be added, 54; this gives 32, which is added to 100, giving 132. This is the sum of the first two quantities. To 132 is added its complement, 68, giving 200. This complement, 68, is subtracted from the third of the quantities to be added, 89, and the difference, 21, is added to 200, giving 221. It is as well here to add the last number, 65, directly to 221, which gives the answer, 286.

In cases where the complement is larger than the number next in order, the difference between the two is subtracted from the hundreds quantity. As an example add 62, 11, 23 and 1.

$62 + 38 + 100$	$38 - 11 = 27$ (to be subtracted)	62
$100 - 27 = 73$	$27 - 23 = 4$ (to be subtracted)	11
$73 + 27 = 100$		23
$100 - 4 = 96$		1
$96 + 1 = 97$		—
		97

The complement, 38, of the first number, 62, is larger than the second number. Therefore the process is reversed and the second number is subtracted from the first complement and the difference is subtracted from 100 instead of being added to it. As the complement of 73, obtained by this subtraction is larger than the third number, 23, the latter is subtracted from 27, and the difference, 4, is subtracted from 100, instead of being added to it. This gives 96, to which we add the last figure of the addition, and thus arrive at the total, 97.

After practicing the operations on paper at full length, there will be no difficulty in doing it almost entirely mentally.

CHAPTER III.

SUBTRACTION

Subtraction

Arithmetical subtraction is the subtraction of a smaller number from a larger one. In algebra the reverse may be done, the smaller quantity may be the minuend, and then the remainder will be affected by a minus sign. But this is outside the scope of arithmetic.

Subtraction may be taken as the simplest of the four elementary processes of arithmetic; ordinarily addition, multiplication or division may be expected to involve more complication. Nevertheless there are a number of modifications of subtraction, a number of ways of doing it, and as we shall see, it may involve complications.

Principles of Subtraction

If we count the cipher, 0, as a digit or number, there will be 100 subtractions of number from number. Thus from 1 we may have to subtract any one of the ten digits: the same is to be said for 2, and as there are ten digits the total number of different subtractions is 10×10 or 100.

In some of these subtractions it is necessary to "borrow ten" and to "carry one"; there are forty five subtractions of single digit from single digit in which this has to be done.

To perform the operation of subtraction these operations have to be known perfectly.

Of ordinary subtraction there is nothing to be said. The smaller number, the subtrahend, is usually placed below the larger, the minuend, and the subtraction is done digit by digit. There is little advantage in a simple subtraction, such as this, in doing the work upon two numbers at once. If it is desired to subtract two digits at once the work can be done exactly on the lines described in the preceding chapter for couple adding.

While the subtrahend is usually placed below the minuend, this is not necessary. A computator should be able to subtract upwards as well and as readily as downwards. It often happens that this has to be done, unless the subtraction is rewritten simply to bring the minuend above the subtrahend. One way should be as easy as the other.

Simplified Subtraction

A simplification of plain subtraction is based on the following considerations. It is easier to subtract a multiple of ten from another quantity, than to subtract any other double digit number. It is easier to subtract 30 than to subtract 27. This is the first consideration. The second one is that if numbers are to be subtracted one from the other, the result will be unchanged if we add the same amount to each or if we subtract the same amount from each. Thus $39 - 27 = 12$. Now add 3 to each and subtract; this gives $42 - 30 = 12$. The same quantity has been added to each and the second subtraction gives the same result as the first. Had we subtracted the same number from each the same result would follow. Subtract 7 from each of the original quantities and subtract the quantities so obtained. This

gives $32 - 20 = 12$, the same remainder as in the other two cases.

Decimalized Subtraction

To utilize these principles we select a number to be added to or subtracted from the minuend and subtrahend, which by such addition or subtraction will produce a multiple of ten as the subtrahend. In the preceding paragraph, it will be observed that this was done. Having added or subtracted the two quantities from minuend and subtrahend alike, we will have an exceedingly simple operation to do to obtain the remainder.

Let it be required to subtract 3865 from 9783. The method could be carried out by adding 135 to each number. This is hardly worth while; the regular way is about as easy. But we may divide the quantities into couples and add 5 to each of the right-hand couples and 2 to each of the left-hand couples, and a very simple operation will give the result, the remainder.

<i>a.</i>	<i>b.</i>		<i>c.</i>
9783	$9783 + 135 = 9918$	$97 + 2 = 99$	$83 + 5 = 88$
3865	$3865 + 135 = 4000$	$38 + 2 = 40$	$65 + 5 = 70$
5918	5918	59	18

The regular method of subtraction is carried out in example *a.*; in *b.* the addition of 135 is employed, and in the third, example *c.*, the subtraction is done in couples, after the addition of 2 and of 5.

The object to be attained is obvious. The subtraction of a decimal number is simple and easy, and such subtraction is brought about by these methods.

Subtraction in Couples

Sometimes when working by couples, there will be one to carry. This introduces no difficulty of any moment. If there is one to carry, 1 is subtracted from the difference of the couple next on the left.

Subtract 3983 from 9765. This is to be done in couples.

Add 1 to the left-hand couples, and add 7 to the right-hand couples. This gives:

9765	98	72
3983	40	90
5782	57	82

In subtracting 90 from 72 we had to borrow 1, so there is one to carry and this is subtracted from the difference of 98 — 40, giving 57 instead of 58.

Sometimes it is better to subtract numbers from quantities of the problem, rather than to add them. If we work in couples, one pair of couples may be treated by subtraction and the other by addition. The last example will be done in couples; in example *a*. 9 will be subtracted from the left-hand couples and 3 from the right-hand couples; in *b*. 1 will be added to the left couples and 3 will be subtracted from the right ones; in *c*. 9 will be subtracted from the left couples and 7 will be added to the right ones.

<i>a.</i>		<i>b.</i>		<i>c.</i>	
88	62	98	62	88	72
30	80	40	80	30	90
57	82	57	82	57	82

In all cases there was one to carry, so that the second subtraction gives 57 instead of 58. In writing out the result the division into couples is dropped—the remainder is given as 5782.

A variation on the above is, as the case may be, to add to or subtract from only one of the quantities. On subtracting a first remainder is obtained; if a quantity was added to the subtrahend, it has to be added to the first remainder; if it was subtracted from the subtrahend it has to be subtracted from the first remainder. If the minuend is the number added to or subtracted from, the first remainder is treated in the reverse way. A single example will suffice to illustrate this method.

Subtract 287 from 3863. Add 13 to 287 giving 300. This is subtracted from 3863 and the 13 is added to the remainder; $3863 - 300 = 3563$ and $3563 + 13 = 3576$. This is a case of adding. Suppose it was 313 to be subtracted from 3576. Taking 13 from the subtrahend leaves 300; subtracting this from 3576 leaves 3276. From this we have to subtract the 13 giving us the final answer 3263.

Subtraction by Inspection

The subtraction of two-figure numbers can be done by decimalization by simple inspection. Thus to subtract 39 from 73, cast out the 9 and the 3 and say 30 from 70 leaves 40. Now the 3 has to be added and the 9 has to be subtracted, giving a net figure of 9 less 3 or 6 to be subtracted from 40, and we have the answer 34. We give three examples, *a*, *b* and *c*.

$$a. \quad 73 - 39 = 40 - 9 + 3 = 34.$$

$$b. \quad 97 - 38 = 60 - 8 + 7 = 59.$$

$$c. \quad 98 - 37 = 60 - 7 + 8 = 61.$$

Subtraction by Addition

When a salesman makes change he frequently makes it by addition. The same process can be applied to any subtraction. Suppose 281 is to be subtracted from 987. The left hand example shows the regular subtraction and

$$\begin{array}{r} 987 \\ 281 \\ \hline 706 \end{array}$$

the right hand one the addition method. To do it first write down 281, the subtrahend. Put a line under it and below this line put down 987, the minuend. Then proceed to write

above the two numbers a number which added to 281 will give 987; this number is 706, and is the answer.

Now subtract 283 from 931 by addition. We proceed as before except that here we have to carry one in the operation twice. Thus having written 283 over the line and 931 under it, we

$$\begin{array}{r} 648 \\ 283 \\ \hline 931 \end{array}$$

we say 3 and 8 are 11. We put down the 8 over the 3 and carrying 1, we say, 8 and 1 are 9 and 4 are 13. The 4 is written above the 8 and 1 is carried to the 2; this gives 2 and 1 are 3 and 3 and 6 are 9. Writing the 6 above the 2 we have the remainder, the answer, 648.

Inverted or Left Hand Subtraction

Inverted or left-hand subtraction is so similar to or, better, analogous to inverted addition, that, if the reader has acquired the one, the other follows. It is the better way to subtract and should always be employed in preference to the usual way.

Subtract 7293138 from 8906872. It is not necessary to write them one above the other; it can be done by simple inspection. We will write it out as usual however, although this is quite unnecessary.

8906872

7293138

 1613734

Proceed thus: 7 from 8 leaves 1—there is nothing to carry as the next two figures to the right subtract without ‘borrowing ten.’ Then 2 from 9 leaves 7, but as the next two figures are 9 from 0, we have to borrow ten—so instead of 7 we write 6. The next figures come regularly until we get to 3 from 7; this is followed on the right by 8 from 2, so instead of 3 from 7 giving 4, as there is 1 to carry we write down 3, and finally write 4 as the remainder of 8 from 12. After a little practice one can write the result off almost as rapidly as if it were being copied. For work in logarithms competent computators do it by simple inspection, writing the result as if copying it.

Subtract 8999 from 9991. Here we look down the line to the right and see that we have in the first and second place to the right, 9 from 9; these involve carrying one because they are preceded by 9 from 1. If we carry 1 to 9 from 9 it gives a remainder of 9 and again 1 to carry. The operation is this: 8 from 9 leaves 1 but as we see that there is 1 to carry the remainder is nothing. 9 from 9 would give nothing but there is 1 to carry, so it becomes 0 from 9 giving 9 as the first figure of the remainder. Then there is another 9 from 9 with 1 carried giving therefore 0 from 9 = 9 as the next figure. We are now at the end and say 9 from 11 gives 2. As we get each figure we write it down—992.

Complement Subtraction

The arithmetical complement of a number is the remainder found by subtracting it from the next highest multiple of 10. It is abbreviated as a. c.

Thus the complement of 1 is $10 - 1 = 9$; the a. c. of 63 is $100 - 63$ or 37. Suppose it is a decimal, then the same rule applies. The a. c. of .361 is $1.000 - 361 = .639$.

The universal way of calculating the a. c. of a number is to subtract its right hand figure from 10 and the others from 9. This avoids the carrying of 1's. The a. c. of 69572 is obtained thus: 6 from 9 gives 3; write this down as the left hand figure. 9 from 9 gives 0; this is put down on the next place to the right; 5 from 9 giving 4 comes next on the right; then 7 from 9 giving 2, and at the extreme right 2 from 10 (not from 9, because it is the right hand end of the row) and the final figure is 8. The a. c. of 69572 is therefore 30428. The next highest multiple of 10 is 10000, or 1 followed by as many ciphers as there are digits in the number. If this is tried once or twice it will be found that the a. c. of a number can be written out just as quickly as the number itself.

The a. c. is used in subtraction; it is especially available where a quantity is to be subtracted from the sum of several others. It is constantly employed in logarithmic calculations.

To subtract by means of the a. c. add the a. c. of the subtrahend and subtract the multiple of ten used in obtaining the a. c. This merely means to throw out a 1 to the left of the a. c. along with any ciphers which come between it and the first number of the sum. To subtract 9872 from 9903 by the a. c. proceed as follows:

	9903
a.c. of 9872	0128
adding	10031
and throwing out 10000	31

The 0 is kept before the a. c. to fix the place of the 1 to be thrown out. The difference or remainder is 31. There is nothing gained by its use in so simple a case. But now take a more complicated case.

Suppose that 1193 is to be subtracted from the sum of 9836, 1072 and 1191. The sum of these three amounts is 12099, and subtracting 1193 from this gives 10906. This involves two operations. To do it by the a. c. write the three numbers to be added one under the other and under them the a. c. of the subtrahend.

	9836
	1072
	1191
a.c. of 1193	8807
adding	20906
throwing out 10000	10906

The throwing out process is simply the subtracting the multiple of 10 used in getting the a. c., or subtracting 1 from the proper decimal place. The proper decimal place is that occupied by the 1 of the multiple of 10 used in getting the a. c; for 1191 we used 10000, so it is 1 in the fifth place which is to be thrown out. Suppose 9991 is to be subtracted from 18931. To get the a. c. of 9991 it is subtracted from 10000. Of course what is really done is to subtract the 1 from 10 and the other figures

from 9 respectively; this gives 9. But the best way to write it is to put a 0 for each of the places where 9 was subtracted from 9, thus 0009, and add it to 18931.

$$\begin{array}{r} 18931 \\ 0009 \\ \hline \end{array}$$

adding and throwing out 10000 8940

which is the remainder or answer.

There are three entries of cash due; \$109.50, \$186.71 and \$199.25; there is one entry of cash owing; \$118.33; calculate the balance by the use of the arithmetical complement.

$$\begin{array}{r} \$109.50 \\ 186.71 \\ 199.25 \\ \text{a.c. of } \$118.33 \\ 881.67 \\ \hline \end{array}$$

adding and throwing out \$377.13 the balance.

Subtraction of Sums

When the sum of several numbers is to be subtracted from the sum of several others, the usual way is to add each set of numbers separately and subtract the sum of one from that of the other set. But it can be done directly. In the example the three numbers below the upper line are to have their sum subtracted from the sum of those above the line. Proceed as follows: Add 2, 2 and 3, giving 7. Keep this in mind and add 9, 9 and 6, giving 24. Subtract 7 from 24 giving 17; put down the 7 under the lower line and carrying 1 add it to the next

$$\begin{array}{r} 9876 \\ 4969 \\ 4989 \\ \hline 1223 \\ 2112 \\ 3212 \\ \hline 13287 \end{array}$$

column above the upper line. This gives 22 from which is to be subtracted the sum of 2, 1 and 1; then 4 from $1 + 8 + 6 + 7$ gives 18; the 8 is put down and the 1 is carried to the next column above the line and the process

9476	is carried through to the end. If there is a
4020	figure in the ten place at the last, as there is
4972	in the example it is put down and the oper-
—	ation is complete. The next example shows
2979	a case where the carrying 1 process applies
1968	to the bottom figures, those of the subtra-
2889	hend. First 9, 8 and 9 are added giving 26;
—	keeping this in mind add 6, 0 and 2 giving 8.

10632 Then 6 (of the 26) is subtracted from 8 and the difference, 2, is put down below the bottom line, and 2 is carried but this time to the lower figures giving $2 + 8 + 6 + 7 = 23$. The figures in the upper column add up 16; 3 (of the 23) from 16 leaves 13; put down 3 and carry the difference between 2 (of the 23) and 1 (of the 16) to the next lower column. It is added to the lower column because the larger number, 2, belongs below the upper line. Following out the same method we next have to subtract 27 from 13; this gives 6 to be put on the lower row, and 2 to carry to the sum of the next lower column and 0 to be carried to the sum of the upper column; the net result is 2 to be carried to the next lower column. Then finally we have to subtract $2 + 2 + 1 + 2$ from the sum of $4 + 4 + 9$, which gives the last or left hand figures of the subtraction, 10, which are accordingly set down.

The process is perfectly simple, the principal point being to keep correct on the carrying; you must carry to the proper place.

Complement Subtraction of Sums

The next method of doing the same kind of subtraction is a variation on the last. The figures in the lower columns

56243

84164

3452

26348

2942

3654

2308

161303

are added and their sums are subtracted from the multiple of ten next above each one. Thus if they add up 14 in any column this is subtracted from 20, because that is the multiple of 10 next above 14. This gives 6, and the 6 is added to the upper column. Suppose the upper column with this addition of 6 adds up 23; then 3 is put down and as there were two tens in the number used to give the com-

plement of the lower column and also the same number of tens in the sum of the upper column there is nothing to carry. Had the lower column required 20 for the subtraction giving the complement, and had the upper column added up 23, then the difference of the tens, i. e. one ten less two tens, would have to be subtracted from the next lower column. Had the tens in the upper column addition exceeded the tens used in getting the complement of the lower column the difference would have to be added to the next lower column.

Referring now to the example, the first lower column adds up 14; its complement is obtained by subtracting 14 from 20 giving 6; this is added to the upper column and the sum is 23; 3 is put down and there is nothing to carry because the 2 of the 23 balances the 2 of the 20, the latter the number required in getting the complement of the lower column addition. The second lower column adds

up 9; its complement $10 - 9$ is 1; this is added to the column above it giving 20; 0 is put down and the difference between 10 and 20 is made unitary by dropping the 0 and as the larger ten figure belonged to the upper column the difference, 1 is subtracted from the third lower column so that its sum becomes 17; its complement is 3, which is added into the upper column giving 13. The difference of complement number of the lower column, 20, and the tens in the sum of the upper column favors the lower column; the difference therefore is added to the fourth lower column, instead of being subtracted as before. This difference is 1; the addition gives 8 as the sum of the fourth lower column, its complement number is 10, and its complement is 2. This is added to the upper column and the sum is 21; 1 is put down and the difference of the tens, 1, is added to the fourth upper column, because there is no lower column to subtract it from. This gives 16, which is set down, and the operation is completed.

A Property of Subtraction

The following is in the line of properties of numbers.

Subtract one number from another; then subtract the second number from the first. In one case you will have to borrow ten, but disregard this and put down the unit figure of the difference. If single figures have been used the sum of the differences will be 10. If double figures have been used the differences added will give 100, and so on.

Some examples follow:

<i>a.</i>		<i>b.</i>		<i>c.</i>	
9	3	96	32	875	221
3	9	32	96	221	675
6	4	64	36	454	546

In example *a.* the remainders add up to 10, in the next example, *b.* the remainders add up to 100, and in the third to 1,000. It will be noticed that the "carrying one" process is left aside.

CHAPTER IV.

MULTIPLICATION

Multiplication a Short Method of Addition

Multiplication is a shortened method of doing addition. If we are to multiply 9 by 7, we at once go to the multiplication table, which everyone is supposed to know by heart, and put down 63. But if the operation is analyzed, it will be found that what has been done is this—seven 9's have been added to each other. Putting it in formula we have: $9 + 9 + 9 + 9 + 9 + 9 + 9 = 9 \times 7 = 63$. Addition of the seven 9's is the same thing in its result as multiplying 9 by 7.

The reverse holds; the addition of the nine 7's is the same in result as multiplying 7 by 9.

The Multiplication Table

The first and all-essential thing in multiplication is to know the multiplication table.

As almost universally taught the table includes as its upper limit, twelve times twelve.

At first sight the multiplication table seems much more formidable than it really is. We have seen that the addition table for two number addition is a very simple affair as the number of sums involved in it are concerned. It contains only 17 sums in 45 combinations.

The multiplication table up to twelve times seems to involve one hundred and forty four operations to be

memorized. On analysis it becomes much simpler. We may omit one times as not to be learned, because it is known to anyone who can count up to twelve. This leaves us one hundred and thirty two operations. Of these eleven are one times; such as 3 times 1 are 3, 4 times 1 are 4; leaving these out we have left one hundred and twenty one operations. But many of these are simple reversals of each other, such as 3×4 and 4×3 . Counting two reversals as one operation, which is perfectly correct, the operations reduce to sixty seven and many products of these operations are repeated, so that the number of products is only forty nine. They are the following:

4	6	8	9	10	12	14	15	16	18	20	21	22	24	25
27	28	30	32	33	35	36	40	42	44	45	48	50	55	
56	60	64	66	70	72	77	80	81	84	88	90	99	100	
108	110	120	121	132	144									

Extending the Multiplication Table

There is no particular reason why the multiplication table should stop at twelve times as it always does. Many teachers advocate its being continued up to twenty times or even twenty five times.

Without absolutely learning all the higher multiplications up to one or the other of these numbers, anyone, doing much calculating, will soon learn many of the multiplications. They can be made available too by taking advantage of the multiplication of the lower known products by 2 or by 4. Fourteen times is seven times multiplied by 2; sixteen times is eight times multiplied by 2 and so for other even multipliers. Then the squares of the higher numbers such as sixteen times sixteen may

be taken as four times sixty four, which last is the square of 8.

But when it comes to the prime numbers, thirteen, seventeen and so on, then there is no simple way, that is really practical, of obtaining them.

It follows that if the learning by heart of the multiplication table for the higher multiplications is to be accomplished, the odd number multiplications should be learned first.

Multiplying by Double or Two Digit Numbers

It is always easier to multiply by a single number than by a double one. Suppose 29 is to be multiplied by 14. If twice 29 is multiplied by half of 14 the answer will be given. So we may proceed thus:

$$29 \times 2 = 58$$

$$14 \div 2 = 7$$

$$406$$

There are any number of variations on this method; the general principle is to multiply or divide by a number which will make a single number out of one of the two given numbers. The trouble is sometimes that one of the numbers may be indivisible without a remainder.

Multiplication of Two Digit Numbers

Quantities of two digits and having the same figure in the tens place, such as 56 and 53, 69 and 64, can be multiplied as follows:

Multiply the units together; multiply the tens digit of one of the numbers by the sum of the units of the original numbers and annex an ought; multiply the tens figures

together and annex two oughts; add the three products for the final results—the product of the original numbers.

Multiply 63 by 69.

$3 \times 9 = 27$. This is the first of the three quantities. Multiply the tens figure, 6, by the sum of the units figures, $3 + 9 = 12$; the product is $6 \times 12 = 72$; annex an ought, giving 720, the second quantity. Multiply the tens figures together, giving 36, and annex two oughts for the third figure, 3600. The sum of the three is the product of 63 by 69. $27 + 720 + 3600 = 4347$.

Two other examples follow.

$$\begin{array}{r} 97 \times 93 = 9021 \\ \hline \end{array}$$

$$7 \times 3 = 21$$

$$9 \times 10 = 900$$

$$9 \times 9 = 8100$$

$$\hline 9021$$

$$\begin{array}{r} 91 \times 92 = 8372 \\ \hline \end{array}$$

$$1 \times 2 = 2$$

$$9 \times 3 = 270$$

$$9 \times 9 = 8100$$

$$\hline 8372$$

The ciphers or oughts have in the above operations been inserted directly, without indication of the operation.

Multiplication of Three Digit and Higher Numbers

This method can be extended to include much higher numbers if the computator knows the squares of large numbers. Two examples of the multiplication of larger numbers follow.

$$\begin{array}{r} 259 \times 257 = 66563 \\ \hline \end{array}$$

$$9 \times 7 = 63$$

$$25 \times 16 = 4000$$

$$25 \times 25 = 62500$$

$$\hline 66563$$

$$\begin{array}{r} 303 \times 308 = 93324 \\ \hline \end{array}$$

$$3 \times 8 = 24$$

$$30 \times 11 = 3300$$

$$30 \times 30 = 90000$$

$$\hline 93324$$

As before the ciphers are inserted or annexed directly.

Increment Multiplication

Suppose there are two numbers of two figures each to be multiplied; by adding an increment to one and subtracting the same increment from the other, we make a decimal number out of one of the two. Then to the difference between the two original numbers add the increment and multiply this sum by the increment, and add it to the first product. The result will be the product of the two original numbers, if the increment was added to the larger and subtracted from the smaller number. But if the increment was added to the smaller and subtracted from the larger, then subtract the increment from the difference between the original numbers and multiply it by the increment, and subtract it from the product of the first multiplication. The first process is carried out in the left hand calculation—the second described one in the right hand one. In both the number 83 is to be multiplied by 62.

$83 + 2 = 85$	21	$62 + 3 = 65$	$21 - 3 = 18$
$62 - 2 = 60$	2	$83 - 3 = 80$	3
5100	23	5200	18
46	2	54	3
5146	46	5146	54

In the left hand example the increment, 2, is added to the larger number and subtracted from the smaller; it is added to the difference between the two numbers, 21, and this sum is multiplied by the same increment, 2, and is added to the product of 85×60 . In the left hand example the increment is 3, it is added to the smaller

number and subtracted from the larger. It is subtracted from the difference between the two numbers, 21, and the result, 18, is multiplied by the same increment, 3, and the product is subtracted from the product of 65 by 80.

The method is interesting but perhaps of little practical value.

Another Method of Increment Multiplication

The next method is a variation on the last one. To each number is added an increment such as will produce a decimal in each case. Then one of the original numbers is multiplied by the other one increased by its increment. This is a simple one figure multiplication. The first number increased by its increment is next multiplied by the increment of the second one, which is also a single figure multiplication; this product is subtracted from the first product. Then the product of the two increments is added to this and is the answer. It is not easy to put into words but it is really simplicity itself as will be seen in the examples.

Let the numbers to be multiplied be 687 by 893. The increments will be 13 for the first number and 7 for the second. The numbers increased by their respective increments will be 700 and 900. They may be arranged as follows:

$$687 + 13 = 700$$

$$893 + 7 = 900$$

Following the rule multiply 687 by 900; this gives 618300. Next multiply 700 by the increment of the other number, 7; this gives 4900. Subtracting we get 613400. To this we add the product of the increments, $7 \times 13 =$

91, which gives the answer, the product of 687 by 893; it is 613491.

We could have gone the other way and multiplied as below.

$893 \times 700 = 625100$. From this subtract the product, $900 \times 13 = 11700 = 613400$ as before. To this add $7 \times 13 = 91$, and we get the result, 613491.

The numbers to be multiplied may reduce to decimals more easily by subtraction of what are really decrements. Proceed exactly as before except that we have to add throughout. Take 805 and 512 as the numbers. The operation follows.

$$805 - 5 = 800$$

$$512 - 12 = 500$$

then $805 \times 500 = 402500$; $800 \times 12 = 9600$; and $5 \times 12 = 60$. Adding the three products gives the product of the original numbers, 412160.

Finally we may add to one number and subtract from the other. Take 812 and 395 as the two numbers to be multiplied. It is obvious that it is a case of adding and subtracting.

$$812 - 12 = 800$$

$$395 + 5 = 400$$

then $812 \times 400 = 324800$; $800 \times 5 = 4000$; $5 \times 12 = 60$; and the sum of the last two products subtracted from the first gives the answer, 320740.

Had we multiplied the other way, 395 by 800, and 400 by 12 and as before 5 by 12, it would have been necessary to add the first two products and to subtract the third. It is advisable therefore to either add increments to both

original numbers or to subtract decrements from both to avoid confusion.

Special Method of Multiplying Three Digit Numbers

The following is an interesting way of multiplying three figure numbers. Multiply 378 by 469. First multiply 378 by 60 this gives 22680. Now multiply 378 by 409 as follows: 9 times 378 are 3402. Write down only the figures, 02, carry the rest in your mind. Then start with 4 times 378. Four times 8 are 32; add this to the 34 and you have 66; write down 6 and carry the other 6; then four times 7 are 28 and 6 are 34; put down 4 and and carry 3; finally four times 3 are 12 and 3 to carry are 15. We now have 154602 to which is to be added 22680, giving 177282 the product asked for.

This method may not be very practical, but this way of multiplying by such numbers as 409, 307 and the like is excellent practice in mental arithmetic.

The method is a special case of cross-multiplication, which latter is given further on in this book.

Multiplication of Special Cases of Polynomials

In multiplication by polynomials it is often convenient to derive one product from another by multiplication or division, instead of recurring to the multiplication of the original multiplicand.

Suppose a number is to be multiplied by 63. After it has been multiplied by 3 in the regular order, instead of multiplying it by 6, the product already obtained may be multiplied by 2, which will give the identical result, as if the original number had been multiplied by 6, because the successive multiplications by 3 and by 2 are the same in effect as multiplying by 6.

We will now set down the multiplication of a quantity by 63.

$$\begin{array}{r}
 3982 \\
 63 \\
 \hline
 11946 \\
 23892 \\
 \hline
 250866
 \end{array}$$

The first product is the result of multiplying the multiplicand by 3; the second product may be obtained by multiplication of the same quantity by 6. But by the method under discussion, it is obtained by multiplying the first product, 11946 by 2.

The operation as written down does not show whether the work is done by the one or by the other method.

Suppose the multiplier had been 36. Then the first product would have been 23892, and the second product would have been obtained by dividing this product by 2, and putting it down one place to the left, just as in regular multiplication. The products and their sum would be

$$\begin{array}{r}
 23892 \\
 11946 \\
 \hline
 143352
 \end{array}$$

This method may be applied when numbers divisible by one another are scattered about among other numbers. Thus suppose a quantity was to be multiplied by 27633. First multiply by 3; copy this down one place to the left for the second product; multiply this product by 2 for

the third product; multiply the original multiplicand by 7 for the fourth product; and finally for the last product divide the third product by 3. An example follows:

$$\begin{array}{r}
 2971 \\
 27633 \\
 \hline
 8913 \\
 8913 \\
 17826 \\
 20797 \\
 5942 \\
 \hline
 82097643
 \end{array}$$

This shows nothing out of the ordinary way, but it will be seen by inspection how it can be done by the processes detailed. Another thing to be observed is that if the multiplier and multiplicand changed places the process could not be applied. Thus set the quantities down thus:

$$\begin{array}{r}
 27633 \\
 2971 \\
 \hline
 \end{array}$$

In this case multiplication can only be done by the regular method, if we take the lower quantity for the multiplier. Of course there is nothing to prevent us from multiplying upwards, using the upper quantity as the multiplier; then the method in question is again applicable.

The method lends itself to many variations; it is an excellent way of proving the correctness of multiplications done by the regular method.

Inverted or Left Hand Multiplication

Inverted multiplication is the term sometimes applied to the following method, where the left hand figures are first multiplied and then the next and the whole then added together.

To multiply 79 by 9 first multiply 70 by 9, which gives 630; then multiply the 9 of the 79 by 9 which gives 81; adding, $630 + 81 = 711$.

Now take a three number quantity; multiply 634 by 8. We have:

$$\begin{array}{r} 600 \times 8 = 4800 \\ 30 \times 8 = 240 \\ 4 \times 8 = 32 \\ \hline \end{array}$$

5072 which is the answer.

Factorial or Proportional Multiplication

Factorial or proportional multiplication is when one of the numbers is multiplied and the other is divided by a quantity which will simplify the operation. Suppose we want to multiply 29 by 14. We can simplify matters by dividing 14 by 2. To carry out the proportion we shall then multiply 29 by the same quantity, 2. This gives $29 \times 14 = 58 \times 7 = 406$.

Take $22\frac{1}{2} \times 36$. We take 4 as the quantity to multiply and divide by. Then $22\frac{1}{2} \times 36 = 90 \times 9 = 810$.

In the last example $22\frac{1}{2}$ was multiplied by 4 and 36 was divided by it. If either of the numbers can be simplified by division we are always free to multiply the other by the same quantity. The reverse is not always the case. Multiplication may simplify one of the numbers where the other is quite intractable to division by the

multiplying number. If 45 were to be multiplied by 27, multiplication of 45 by 2 would give 90, a very simple number to multiply by, but we cannot divide 27 by the same 2. But this difficulty may be met by the following method.

If multiplication will give a simpler multiplier, multiply the multiplier, and then do the final multiplication, and divide the product by the number used to simplify one of the quantities. Let us take 431×45 .

Multiply 45 by 2; this gives 90, an easy number to multiply by. $431 \times 90 = 38790$. Divide this by 2 and we have 19395, which is the product of the two original numbers, $431 \times 45 = 19395$.

Under this process fall a quantity of numbers; namely those ending in 5 up to 55 give multipliers within the range of the multiplication table.

Multiply 269 by 55. Doubling 55 gives 110. $269 \times 110 = 29,590$, and half of this is 14,795.

As an alternative in this case we would equally as well have multiplied by 11 and by 5 in succession.

Aliquot Part Multiplication

An aliquot part of a number is a number which can be a divisor of the number without any remainder. It is a factor. Thus 5 is an aliquot part of 15 or of 25, because either of the numbers in question can be divided by 5 without any remainder; it is a factor of either number.

Aliquot parts of 100 are often of much use in abbreviating operations; suppose we want to know the sixteenth part of 2000; from aliquot parts we know that the sixteenth of 100 is $6\frac{1}{4}$; therefore the sixteenth part of 2000

is 125, for that is the product of $6\frac{1}{4}$ by 20, and $100 \times 20 = 2000$.

In the statement of aliquot parts we simply give the equivalents of aliquot parts of 100 as fractions. Each aliquot part is given as a fraction of 100.

2..... $\frac{1}{50}$	$12\frac{1}{2}$ $\frac{1}{8}$	60..... $\frac{3}{5}$
4..... $\frac{1}{25}$	$13\frac{1}{3}$ $\frac{2}{15}$	$66\frac{2}{3}$ $\frac{2}{3}$
5..... $\frac{1}{20}$	$16\frac{2}{3}$ $\frac{1}{6}$	75..... $\frac{3}{4}$
	20..... $\frac{1}{5}$	
$6\frac{1}{4}$ $\frac{1}{16}$	25..... $\frac{1}{4}$	80..... $\frac{4}{5}$
$6\frac{2}{3}$ $\frac{1}{15}$	$33\frac{1}{3}$ $\frac{1}{3}$	$83\frac{1}{3}$ $\frac{5}{6}$
$8\frac{1}{2}$ $\frac{1}{12}$	50..... $\frac{1}{2}$	$87\frac{1}{2}$ $\frac{7}{8}$

As aliquot parts refer to 100, they are really hundredths, and the left hand columns with a cipher preceding them, can be written as decimals—.02, .04 &c. until we get to the double figure numbers; these with the decimal point read at once as hundredths, thousandths &c. without any cipher preceding them—.12 $\frac{1}{2}$ or more properly .125, .1333..., &c.

As these aliquot parts refer to 100 parts as their basis, they are useful in calculations involving dollars; the third of a dollar is $33\frac{1}{3}$ cts. the fifth of a dollar is 20 cts.

The above set of parts is far from complete, for there are any number of aliquot parts, which may be of use, with various bases. Once the idea of how to use them is grasped their use may be greatly extended according to the operations to be done.

Aliquot parts then are of constant use in multiplications. The following examples will show how they can be used.

To multiply by 50 annex two ciphers and divide by 2.
 $32 \times 50 = 3200 \div 2 = 1600$.

To multiply by 25 annex two ciphers and divide by 4.
 $28 \times 25 = 2800 \div 4 = 700$.

To multiply by 20 annex two ciphers as hitherto and divide by 5. In this case it would generally be easier to multiply directly, but there may be reasons for doing it in the indirect way.

The operations may be combined. Thus to multiply by 75, annex two ciphers and divide first by 4; then divide the original number with the two ciphers by 2 and add the quotients. $29 \times 75 = 2900 \div 4 = 725$ and $2900 \div 2 = 1450$ and $725 + 1450 = 2175$.

Practical Use of Aliquot Parts in Multiplication

One of the conveniences of the use of aliquot parts is that they enable us to dispense with fractions. They are only applicable, as given here, to hundreds and other decimal numbers, but as our coinage is decimal they have extensive application in business calculations. If the metric system is being used, then again they may be useful.

Some examples of their applications follow.

What is the cost of 55 articles at \$2.50 each? Add a cipher and divide by 4; $550/4 = \$137.50$.

Multiply 63 items by \$2.12½. Multiply 63 by 2, and add 63/8, which is 7¾ or \$7.87½; this gives \$133.87½.

27 yards at 62½ cts. is to be charged. This is 5/8 of \$1.00; as $5/8 = 1/2 + 1/8$ we add together 27/2 and 27/8, which gives $13.50 + 3.37½ = \$16.87½$.

Multiply 28791 by 125. This may be done in either of two ways. We may add three ciphers and divide by 8; this is because the list of aliquot parts tells that 125 is 1/8 of 1000; this gives 3598875. Or we may multiply

by 100 and add one quarter of the product thereto, because 25 is one quarter of 100; $2879100 + 719775 = 3598875$ as before.

Suppose it is asked what the product of 1000 by $\frac{5}{8}$ is. This is given directly by aliquot parts; it is 625. Or to 500, which is $\frac{4}{8}$ of 1000, may be added $\frac{1000}{8}$, or $\frac{1}{8}$ of 1000, this is 125, and adding the two gives 625 as before.

There is another thing to be said about aliquot parts, and this applies to many fractional calculations, where a mixed number occurs. It may be told by examples.

Multiply 100 by $2\frac{1}{4}$. This is done in one way by multiplying the multiplicand, 100, by 2 and then by $\frac{1}{4}$ and adding the two for the answer. Thus we have $100 \times 2 = 200$; $100 \times \frac{1}{4} = 25$; $200 + 25 = 225$. This is simple but there is another way of doing it, which may be more convenient in some cases. The original number may be multiplied by the integral number of the multiplier, and then the product may be multiplied by one half the fraction and the two are then to be added together. In the case just used, 100 would be multiplied by 2, giving 200, and to that would be added $\frac{200}{8}$, which is 25, and the result will be 225, the same as before.

If the multiplier had been $4\frac{1}{2}$, then to the first product, 400, there should have been added $\frac{1}{16}$ of 400. The result is the same of course in all methods.

This method is of assistance in some fraction work. Suppose a number is to be multiplied by $2\frac{2}{3}$. It is manifestly easier to multiply the number by 2 and add to the product $\frac{1}{3}$ of the product than it is to add to the same product $\frac{2}{3}$ of the original number. For to multiply by $\frac{1}{3}$, we have only to divide by 3; to multiply by $\frac{2}{3}$ we have to multiply by 2 as well as to divide by 3; one is twice the work of the other.

Mixed numbers, not exactly within the scope of this treatment, may often be brought within it. Suppose it is a question of multiplying some number by $6\frac{3}{4}$; this mixed number can be written $6\frac{6}{8}$, for $\frac{6}{8} = \frac{3}{4}$. Then to multiply by it, the multiplicand is multiplied by 6 and there is added to the product $\frac{1}{8}$ of the product. It is evident that $\frac{1}{8}$ of the product is equal to $\frac{6}{8}$ of the multiplicand. It is easier to do this than to proceed in the regular way and multiply the original number, the multiplicand, by 3 and divide the result by 4 and thus obtain the quantity to be added to the first product.

Multiply 77 by $8\frac{2}{5} = 8\frac{8}{20}$. Multiplying 77 by 8 gives 616. To this add $616 \times \frac{1}{20}$, or what is the same thing, divide 616 by 20 and add. This gives $616 + 30.80 = 646.80$. In the regular way we should have multiplied 77 by 2 and divided by 5 to get the quantity to be added to 615. The first way is the easier.

Aliquot parts, as given here, refer to 100 as the base. The principle involved in their use may be extended by factor multiplying. This method is of such extensive and of so many special applications, that it can best be given by means of examples and by the statements of special cases. The computator will have no difficulty in applying it to a new case, once the idea is grasped.

Factor Multiplying

To do factor multiplying one of the numbers to be multiplied is factored; this one is taken as the multiplier and the other one is multiplied by the factors in succession.

To multiply by any even number between 12 and under 25. Divide the multiplier by 2; multiply the other number by this quotient and then multiply this product by 2.

Multiply 1986 by 18. 18 divided by 2 gives 9. $1986 \times 9 = 17,874$; $17,874 \times 2 = 35,748$, the answer. The first multiplication could have been by 2 and the second by 9.

A number of multipliers, up to 36 in value can be brought within the range of the multiplication table by division by 3.

Multiply 3986 by 36. In accordance with the principle we divide 36 by 3; this gives the two factors 3 and 12; successive multiplication by these gives the product; $3986 \times 12 = 47,832$ and $47,832 \times 3 = 143,496$.

Division by 4 carries the principle up to 48 for numbers divisible by 4. In this way we can go as high as 108 for numbers divisible by 9. By using three successive multiplications by three factors of the multiplier the method can be greatly extended; still greater numbers of factors can be used until the limit of usefulness is passed.

Thus it is as easy to multiply by 243 directly, as it is by 3, by 9 and by 9 again in succession. These are three factors of 243 which could be used for factor multiplying.

This method applies well to numbers ending in one cipher or in several ciphers. To multiply by 180, use as successive multipliers 2 and 90, or 3 and 60.

We have seen that to multiply by 25 we may add two ciphers and divide by 4. Suppose a number is to be multiplied by 24; it is clear that the two ciphers may be added and the division by 4 will give the product one time too high. Therefore all that is necessary is to multiply by 25; this is done by adding two ciphers and dividing by 4; then subtract the number. Suppose 243 is to be multiplied by 124. First multiply by 100; this takes care of the initial 1. Then add two ciphers, divide by 4 and add

it to the product just obtained. Finally subtract 243 and you will have the correct result. The operation follows:

$$\begin{array}{r}
 243 \times 100 = 24300 \\
 24300 \div 4 = 6075 \\
 \hline
 \text{subtract } 243 \quad 30375 \\
 \quad \quad \quad 243 \\
 \hline
 30132
 \end{array}$$

If 26 were the multiplier, the multiplicand would be added instead of subtracted.

To Multiply by Nine

To multiply a number by 9, write the number down and imagine a cipher placed after it. Then subtract the number from the unwritten one. It amounts to subtracting the right hand figure of the original number from 0, carrying one and subtracting the next figure of the original one increased by the 1 carried from the first figure and so on. Thus in the example we say 3 from 0 leaves 7 and 1 to carry; adding this 1 to the second figure of the original, 8, gives 9, and 9 from 3 gives 4 with 1 to carry. Carrying the one to the next figure, 5, we have 6 from 8 leaves 2 with nothing to carry. 7 from 5 gives 8 with 1 to carry; carrying the 1, 7 from 7 leaves 0 with nothing to carry, so finally the 6 is put down completing the multiplication. Had there been 1 to carry at the last the 6 would have been 5 in the product at the left hand.

67583

608247

To Multiply by Eleven

To multiply by 11 write down the number. Then start by putting down its right hand figure as the first figure of the product. Then add the second figure to the first and put down the right hand figure of the sum if there are two figures in it and if there are carry 1. If there is only one figure put that down, there being no 1 to carry. Add the third figure to the second adding 1 if it is to be carried and write the first figure down as before. In the example 3 is put down; then $8 + 3$ gives 11; 1 is put down and 1 is carried. Then $1 + 5$ are 6 and $6 + 8$ are 14; the 4 is put down and 1 is carried. $1 + 7$ are 8 and $8 + 5$ are 13; 3 is put down and 1 is carried. $1 + 6$ are 7 and $7 + 7$ are 14; 4 is put down with 1 to carry. This is added to the final figure, 6, and 7 is written down as the right hand figure of the product.

67583

743413

To Multiply by One Hundred and Eleven

To multiply by 111 the operation is the same essentially as the one last described. To multiply the same number by 111 first put down the 3 for the right hand number; then add 8 and 3 giving 11 and put down 1; then add the 1 to be carried to $5 + 8 + 3$ and put down 7 and carry one; then $1 + 7 + 5 + 8$ put down 1 and carry 2, and so on.

Complement Multiplication

Complement multiplication may be done in a simple way by taking into the operation the true complements of the numbers, and may be made to include numbers varying in the ten places.

Subtracting each number from 10, 100 or other power of ten gives its complement. Multiply the complements together and add to the product the difference between either number and the complement of the other multiplied by 100.

Two examples are given, one of numbers with the same ten figures, the other with figures having different ten figures.

Multiply 97 by 93.

Complements are 3 and 7.

$$\begin{array}{r} 3 \times 7 = 21 \\ 97 - 7 \text{ or } 93 - 3 = 90 \\ 90 \times 100 = 9000 \\ \hline 9021 \end{array}$$

Multiply 87 by 95.

Complements are 13 and 5.

$$\begin{array}{r} 13 \times 5 = 65 \\ 87 - 5 \text{ or } 95 - 13 = 82 \\ 82 \times 100 = 8200 \\ \hline 8265 \end{array}$$

It makes no difference which number is diminished by the complement of the other as the result is the same.

The next example shows the same operation applied to three digit numbers. Here the complement is the difference between the number and 1000, and the difference of the complement of one of the numbers and the other number has to be multiplied by 1000 instead of by 100.

Multiply 931 by 972.

The complements are 69 and 28.

$$931 - 28 \text{ or } 972 - 69 = 903.$$

$$\begin{array}{r} 69 \times 28 = 1932 \\ 903 \times 1000 = 903000 \\ \hline 904932 \end{array}$$

Multiplication of Numbers Ending with 5

To multiply two numbers together, each of two digits and both ending in 5, proceed as follows: Put down 25 in the right hand place; multiply together the figures to the left of the fives and put the product next to the 25 on the left; then add half the sum of the figures to the left of the fives, taking them in their proper decimal place,

Multiply 45 by 25. Put down 25 to the right; multiply 4 by 2 giving 8; to this add half the sum of 4 and 2, which added to the 8 already found gives 11, which is put to the left of the 25; the result is 1125.

Multiply 35 by 45. Here the sum of the tens is uneven. Proceeding as before put down 25; put in front of it the product of the tens and add half their sum; the result is 1575. The operation is done as shown here:

$$\begin{array}{r}
 5 \times 5 = 25 \\
 30 \times 40 = 1200 \\
 30 + 40 = 70; 70/2 = 35 \\
 \hline
 1575
 \end{array}$$

Multiplying by Two Numbers at Once

To multiply by two numbers at once multiply the two figures of the multiplier by each figure of the multiplicand, writing down the last figure and carrying the one or more left hand figures. There is no object in doing it, if the whole operation has to be done at length. In the example the work placed on the right should be done mentally in practice.

$$\begin{array}{r}
 1575 \quad 5 \times 23 = 115 \\
 23 \quad 7 \times 23 = 161 \\
 \hline
 36225 \quad 5 \times 23 = 115 \\
 \quad 1 \times 23 = 23 \\
 \hline
 36225
 \end{array}$$

Next as a development of the process we will put down the four products obtained, with the numbers to be carried added in ; then taking the last figures of the set and putting the last left hand figure obtained in front we again get the result. If the method is applied these four products will be: 115, 172, 132, 36, and the last digits in reverse order, with the 3 of the 36 last obtained in front, gives the answer or product, as before.

Multiplying by Any Number between Twelve and Twenty

To multiply by any number between 12 and 20, the following method is of interest.

Multiply the figures of the multiplicand in the regular order one by one by the unit figure of the multiplier. If there is anything to carry add it as usual, and add also to each product thus obtained the figure of the multiplicand on the right of the figure multiplied. Put down the right hand figure of the number thus obtained and carry the left hand figure if there is one. The tens figure of the multiplier does not enter into the calculation directly ; it is taken care of in the process as described.

An example follows, with the successive steps in detail.

$$\begin{array}{r}
 39712 \\
 17 \\
 \hline
 675104
 \end{array}$$

$7 \times 2 = 14$. Put down 4 and carry 1. *b.* (7×1)
 $+ 1 + 2 = 10$. Put down 0 and carry 1. *c.* (7×7)
 $+ 1 + 1 = 51$. Put down 1 and carry 5. *d.* (7×9)
 $+ 5 + 7 = 75$. Put down 5 and carry 7. *e.* (7×3)
 $+ 7 + 9 = 37$. Put down 7 and carrying 3 add it to the
 left hand figure of the multiplicand and put down the
 sum as the last figure.

In carrying out this method it is important to attend to the addition of the left hand figure to the last figure carried.

“Multiplying by the ‘Teens’”

This operation, when the multiplier is below 20 in value and exceeds 10 is called “multiplying by the teens.” It is a modification of cross-multiplication, to the extent that cross-multiplication includes the possibility of any sized multiplier. Its use is only limited by the ability of the computator. Cross multiplication is an excellent test object to determine a computator’s powers. If the multiplier is a large number it is very difficult to cross-multiply successfully.

Cross-Multiplication

Cross-multiplying is a method of multiplying by a number or quantity of more than one digit, without putting down the partial products. It will be best explained by examples. The following principles apply to its execution.

Call the right hand digit of any quantity the first number of the quantity; the next digit to the left the second and so on. Then if a quantity is multiplied by another, any single number or digit of the multiplicand multiplied by the first digit of the multiplier, will have

the first figure of this product directly under itself. If multiplied by the second digit the first figure of the product will be shifted one place to the left; if multiplied by the third digit it will have its first figure shifted two places to the left. If the product of a digit contains two figures, the second one will fall into its regular place to the left of the first one. We will now proceed to give some examples.

Multiply 72 by 63. Call 63 the multiplier.

Then: $2 \times 3 = 6$, the first figure of the product of the two original numbers. Next: $2 \times 6 = 12$ added to 7×3 gives $12 + 21 = 33$ and 3 is the second number of the product. We have therefore 3 to carry. Finally $7 \times 6 = 42$ and carrying the 3 gives 45, the third and fourth figures of the product, which is 4536.

Multiply 81 by 37, calling 37 the multiplier. It is done in formulas.

81×37 . $7 \times 1 = 7$; the first number of the product.
 $(7 \times 8) + (3 \times 1) = 59$; 9 is the second number with
 5 to carry.

$(8 \times 3) + 5 = 29$; the third and fourth figures of
 the product.

The entire product is 2997.

Multiply 736 by 84; the operations are in formulas.

736×84 . $6 \times 4 = 24$; 4 is the first figure; 2
 to carry.

$(6 \times 8) + (3 \times 4) + 2 = 62$; 2 the second figure; 6
 to carry.

$(3 \times 8) + (7 \times 4) + 6 = 58$; 8 the third figure; 5 to
 carry.

$(7 \times 8) + 5 = 61$; the fourth and fifth
 figures.

The entire product is 61824.

It will be observed that every figure in the multiplicand is multiplied by every figure in the multiplier, and that each digit of the products falls into its proper place.

Multiply 429 by 643; the operations are in formulas.

$429 \times 643.$ $3 \times 9 = 27$; 7 the first figure; 2 to carry.

$(9 \times 4) + (2 \times 3) + 7 = 44$; 4 the second figure; 4 to carry.

$(9 \times 6) + (2 \times 4) + (4 \times 3) + 4 = 78$; 8 the third figure; 7 to carry.

$(2 \times 6) + (4 \times 4) + 7 = 35$; 5 the fourth figure; 3 to carry.

$(4 \times 6) + 3 = 27$; the fifth and sixth figures.

The entire product is 275,847.

Multiply 3987 by 4926. The operations are in formulas.

$3987 \times 4926.$ $7 \times 6 = 42$;
2 the first figure; 4 to carry.

$(7 \times 2) + (8 \times 6) + 2 = 66$;

6 the second figure; 6 to carry.

$(7 \times 9) + (8 \times 2) + (9 \times 6) + 6 = 139$;
9 the third figure; 13 to carry.

$(7 \times 4) + (8 \times 9) + (9 \times 2) + (3 \times 6) + 13 = 149$;
9 the fourth figure; 14 to carry.

$(8 \times 4) + (9 \times 9) + (3 \times 2) + 14 = 133$;
3 the fifth figure; 13 to carry.

$(9 \times 4) + (3 \times 9) + 13 = 76$;

6 the sixth figure; 7 to carry.

$(3 \times 4) + 7 = 19$;

the seventh and eighth figure.

The entire product is 19,639,962.

Cross-multiplication is the *ne plus ultra* of this operation; if you can do it with a four or five digit multiplier and an equally long or longer multiplicand, you may consider yourself a proficient. It has been explained by examples; after these are gone through carefully, the principle will be understood. It is hard to give a verbal rule of any value for the process.

In using it the operations should not be put down; nothing but the result should appear, figure by figure. By practice great expertness can be attained. Some can do it with numbers of twelve or more digits.

Slide Multiplication

Cross-multiplication can be done in another way called the sliding method. The multiplier is written out on a separate slip of paper but in inverted or reversed order. It is placed over or under the multiplicand and is shifted constantly one place to the left in succession after the multiplications indicated by its positions are done. We will repeat some of the examples already done by the regular method of cross-multiplication, doing them by the sliding method.

Multiply 72 by 63. Write the multiplier reversed upon a slip of paper, thus: 36. Place it over the multiplicand with its left-hand figure over the right-hand figure of the multiplicand, and multiply as indicated.

36

72 $3 \times 2 = 6$; the first figure of the product.

Now slide the paper slip one place towards the left; two multiplications will now be indicated.

36

72 $(6 \times 2) + (3 \times 7) = 33$; put down 3, the second figure; 3 is to carry.

Next slide the paper slip one more place to the left and multiply as indicated.

36

72 $(6 \times 7) + 3 = 45$; the third and fourth figures of the product.

The entire product is 4,536.

Multiply 429 by 643. Write 346 on the slip of paper, and proceed as in the last example.

346

429 $3 \times 9 = 27$; 7 the first figure; 2 to carry.

The slip of paper is shifted one place to the left.

346

429 $(9 \times 4) + (2 \times 3) + 2 = 44$; 4 the second figure 4 to carry.

The paper is shifted one more place to the left.

346

429 $(9 \times 6) + (2 \times 4) + (4 \times 3) + 4 = 78$; 8 the third figure; 7 to carry.

The paper is shifted one more place to the left.

346

429 $(2 \times 6) + (4 \times 4) + 7 = 35$; 5 the fourth figure; 3 to carry.

The paper is shifted to the left for the last time.

346

429 $(4 \times 6) + 3 = 27$; the fifth and sixth figures.

The entire product is 275,847.

After sufficient practice the paper slip can be dispensed with; the multiplier can be written in reverse order over or under the multiplicand and the work can be performed mentally, so that nothing is done on paper except the writing down the answer, digit by digit. There is not the least difficulty in doing it, but it is well to practice with the slip of paper first.

Excess of Nines Proof of Multiplication

The excess of nines method of testing the accuracy of multiplication is of special interest as an application of one of the properties of the number nine. If the digits of a number are added together and their sum is divided by 9 the remainder if there is any, is called the excess of nines. Instead of adding all the digits the best way of getting the excess is to do as in the example.

Required the excess of nines in 192846. Proceed as follows: 9 and 1 are 10, an excess of 1. Add this to the next digits; 1 and 2 and 8 are 11, an excess of 2; then as before $2 + 4 + 6 = 12$, an excess of 3. The quantity, 3, is reached by subtracting 9 from the last sum, 12, or it may be found by adding the digits of 12 together, $1 + 2 = 3$.

If a multiplication of two quantities is correctly performed, the excess of nines in the multiplicand multiplied by the excess in the multiplier and the excess in this product taken will equal the excess of nines in the product of the multiplication.

Test the accuracy of the following multiplication by excess of nines: $39821 \times 8769 = 349,190,349$.

The excess of nines in the first number, the multiplicand, is 5. The excess in the second number, the multi-

plier, is 3; the product is 15, and the excess of nines in 15 is 6. The excess of nines in the product of the original multiplication is also 6, so the operation comes out right by the test. It is to be remembered that this is only a test, not an absolute proof. If the excess comes out differently then the operation is certainly wrong.

Curiosities of the Multiplication Table

There are a number of curious things about the multiplication table, which things are of more interest than utility. Some of them will be given here.

If the products of any of the divisions of the table, such as three times or four times, are written down, and if their unit figures are added up, the sum of the figures in question, stopping at the ninth place, will be either 40 or 45 except in the case of five times. For even number multiplications such as four times or six times the sum of these units will be 40; for the odd times, three times or seven times, the sum will be 45. A number of examples are given below:

Three times: 3	Four times: 4	Six times: 6	Seven times: 7
6	8	12	14
9	12	18	21
12	16	24	28
15	20	30	35
18	24	36	42
21	28	42	49
24	32	48	56
27	36	54	63
—	—	—	—
45	40	40	45

The left-hand columns are not added up; the right-hand columns are the only ones added. The five times section is not put down; this gives as the sum of its right-hand digits only 25. If now the sum of the odd numbers in any uneven number section, such as three times or seven times are added, their sum will be 25. It is done here for three times, seven times and nine times:

Three times 3	Seven times 7	Nine times 9
9	21	27
15	35	45
21	49	63
27	63	81
—	—	—
25	25	25

It is, as before, only the right-hand numbers which have been added up, and their sum is the same as the sum of all the right-hand digits of the five times division of the multiplication table.

Now take any of the columns and add, this time horizontally, the component digits of the different products; taking the three times column given above, its digits add up thus: 3, 6, 9, 3, 6, 9, 3, 6, 9. The next column adds up thus: 4, 8, 3, 7, 2, 6, 10, 5, 9. The next column, six times, gives: 6, 3, 9, 6, 3, 9, 6, 12, 9, and if the digits of the anomalous looking 12 are added together they give the missing 3.

Various degrees of regularity can be traced out for these additions; the tracing and following them out may be left to the reader. Doing this constitutes a sort of "Arithmetic Solitaire."

Write down the nine digits in a vertical column, be-

ginning with 1 and ending with 9. Then to the right of
 9 this column write another one made up of the
 18 same nine digits, beginning this time with 9,
 27 one space above the 1 of the other column;
 36 by the side of the 1 of the first column place
 45 the 8 of the new column and so until all nine
 54 figures are written out. Then the full nine
 63 times of the multiplication table will be seen
 72 from nine times one up to nine times nine.
 81 The two columns are given at the side of the
 page.

A Curious Method of Multiplying

It is possible to multiply any numbers together and use only simple addition, multiplication by 2 and division by 2.

Put the two numbers down side by side. Divide one of them by 2, put the quotient under the same number and divide this quotient by 2. Pay no attention to remainders. Repeat this until you can go no further, or until the quotient 1 is obtained. Multiply the other number by 2, put it alongside of the first quotient, multiply this by 2 and put it alongside of the second quotient. Keep this up until you have a multiple for each of the quotients. The quotients can go only a definite distance and are the limiting element. Of the products thus obtained strike out each one that is opposite an even number quotient; the sum of the products remaining will give the product of the two original numbers.

Multiply 68 by 91.

It is immaterial which number is successively divided and which multiplied. It is done in both ways here, indicated as *a* and *b*.

<i>a</i>	<i>b</i>
6891	68 91
34182	136 45
17364	272 22
8728	544 11
41456	1088 5
22912	2176 2
15824	4352 1

The quantities opposite the odd number quotients have to be added to give the product of the two original numbers. This is done below in each case below its own calculation.

364	68
5824	136
	544
6188	1088
	4352
	6188

Oddities in Multiplication

If the digits with the omission of 1 are written in reverse order and multiplied by 9, the product will be a succession of nine 8's.

$$98765432 \times 9 = 888888888$$

Now write the nine digits including the 1 this time, and multiply by 9 and the product will be the nine 8's as before with a 9 on the right end. If multiplied by 18, the left hand figure of the product will be a 1, then will come nine 7's and as the right hand figure there will be an 8.

If multiplied by 27, the left hand figure will be a 2, the nine middle figures will be 6's, and the right hand figure a 7. It goes thus through the multiples of 9 used as multipliers, until we finally get for nine times nine or 81 as a multiplier, the left hand figure, 8, the right hand figure, 1 and ciphers to the regular number of nine, as the intermediate digits.

In each product the left and right hand figures give or repeat the multiplier, and the intermediate figures run in regular order from 8's to ciphers. Some of the multiplications are given here:

$$\begin{array}{rcl}
 987654321 \times 9 & = & 8888888889 \\
 \text{ditto} \times 18 & = & 1777777778 \\
 \text{ditto} \times 27 & = & 2666666667 \\
 * & * & * & * & * & * & * \\
 * & * & * & * & * & * & * \\
 \text{ditto} \times 81 & = & 8000000001
 \end{array}$$

Of course for the left hand figure of the product of the multiplication by 9, there is no left hand figure to be supplied; the final or right hand 9 gives the multiplier.

The number 15873 multiplied by 7 gives as product six 1's.

$$15873 \times 7 = 111111$$

Now multiply this same number by 9 and the product is 142857, and this number multiplied by 7 gives as product six 9's.

$$15873 \times 9 = 142857 \quad \text{and} \quad 142857 \times 7 = 999999$$

or

$$15873 \times 63 = 999999$$

The following is an odd series of products. As usual it is the number 9 that is the main factor in the operations.

$$9 \times 9 = 81 \text{ and } 81 + 7 = 88$$

$$9 \times 98 = 882 \text{ and } 882 + 6 = 888$$

$$9 \times 987 = 8883 \text{ and } 8883 + 5 = 8888$$

The last two in the series are:

$$9 \times 9876543 = 88888887 \text{ and } 88888887 + 1 = 88888888$$

$$9 \times 98765432 = 888888888 \text{ and } 888888888 + 0 = 888888888$$

The number 153846 multiplied by 13 produces 1 as the left hand digit, 8 as the right hand digit, with five 9's between them.

$$153846 \times 13 = 1999998$$

In many cases these curious multiplications can be carried out by further multiplications, so as to give other results. We have seen that if we multiply 153846 by 13 the product is 1999998. If we add one half of itself to the above number, namely 76923 we obtain 230769, and this multiplied by 13 gives 2999997. Adding it again to the last sum we obtain 307692, and multiplying this by 13 we get 3999996. This process can be carried down to the eighth product, which will be 153846 multiplied by 5, for that is what the successive additions of 76923 leads to; the product of 153846 by 5 is 769230 and the product of this number by 13 gives 9999990. Now these products may be placed in columns; the first three are given in full.

$$153846 \times 13 = 1999998 \quad 5999994$$

$$230769 \times 13 = 2999997 \quad 6999993$$

$$307692 \times 13 = 3999996 \quad 7999992$$

$$384615 \times 13 = 4999995 \quad 8999991$$

The left hand digits run from 1 to 8; the right hand ones run from 8 to 1; the left and right hand digits give the products of 9 by 2, 3 &c.

Other odd multiplications are the following:

$$37037037037 \times 9 = 33333333333$$

$$13717421 \times 9 = 123456789$$

$$987654321 \times 9 = 888888889$$

Finger Multiplication

The following way of multiplying two numbers each more than 5 and less than 10, is more curious than useful, for everyone is supposed to know the multiplication table.

Suppose we are to multiply 9 by 8. Hold up the fingers of both hands open. Take the difference between one of the numbers and 10 and bend down a finger or fingers corresponding in number to that difference. Do this on one hand for one of the numbers and on the other hand for the other number. The fingers not bent down give the tens and the product of the fingers bent down on one hand by those on the other hand give the units. In the case of 9 and 8 two fingers will be bent down on one hand and one on the other; their product is 2; there are seven fingers left standing; these are the tens; putting them before the 2 we have 72, the answer.

This operation can be carried out very well on an abacus.

If the fingers to be held down exceed the number 10, subtract the excess from the tens of the product.

CHAPTER V

DIVISION

Factor Division

If a number is to be divided by one so large that it requires long division, if the divisor can be factored down far enough a series of short divisions by the factors can be substituted for the long division.

In the example 30672 is divided by 432. The latter can be factored thus: $12 \times 12 \times 3$, and three short divisions by these factors give the quotient, 71.

$$\begin{array}{r} 12 \overline{) 30672} \\ 12 \overline{) 2556} \\ 3 \overline{) 213} \\ \hline 71 \end{array}$$

This division has no remainder. Let it be required to divide 34577 by 18; here there is a remainder. Divide

by the factors, 2, 3, and 3. Here there are three remainders; each remainder is to be multiplied by the factors of the divisions preceding its own and the sum of the products is the total remainder. The first remainder is not multiplied by anything; this is under the rule for no division precedes it.

$$\begin{array}{r} 2 \overline{) 34577} \\ 3 \overline{) 17288} \quad 1 \dots 1 \\ 3 \overline{) 5762} \quad 2 \dots 5 \\ \hline 1920 \quad 2 \dots 17 \end{array}$$

The remainder from the division of the whole number, 34577 by the first factor, 2, is 1. The remainder from the next division, which is the division of half the number by 3 is 2. This is multiplied by the first divisor, 2, and is added to the first remainder, 1, giving 5. The next

remainder comes from the division of $\frac{1}{6}$ th the number by 3. This remainder is multiplied therefore by 3 and by 2, or by 6, and is added to 5 just determined, giving 17 which is the total remainder.

Or start at the bottom and multiply the last remainder, by 3 and add it to the next remainder above it; this gives $(2 \times 3) + 2 = 8$. Multiply 8 by 2 and add the first remainder, 1, and the total remainder is obtained, namely 17.

Abbreviated Long Division

To abbreviate to a certain extent the work of long division the writing out of the products may be omitted, and the subtractions may be made mentally and the remainders only written down.

Divide 27815 by 31.

It is put down as for long division. $31 \overline{) 27815} (897\frac{8}{31}$
 Inspection shows that the first figure of the quotient is 8. Then we say 8 times 1 are 8 and subtracting this first figure of the product from the dividend we put down 0 with nothing to carry. Next 8 times 3 are 24, which subtracted from 27 leaves 3 which is put down. The 1 is taken down mentally from the dividend and it is seen that 31 goes into 301 9 times. Proceeding we have 9 times 1 are 9 and subtracting this 9 from the 1 of the dividend leaves 2 which is put down with 1 to carry. Then 9 times 3 are 27 and 1 to carry makes 28 which subtracted from 30 leaves 2, the last figure of the remainder. Taking down 5 mentally, 31 goes into 225 7 times. 7 times 31 are 217, which subtracted from 225 leaves a final remainder of 8, which is written in the quotient as numerator of a fraction $\frac{8}{31}$.

Italian Method of Long Division

What is termed the Italian method of doing long division consists in putting both divisor and quotient on the right hand of the dividend. It is just as good to put them on the left hand. The example shows the method. The advantage is that the divisor and quotient are in position to be multiplied to prove the correctness of the operation. The operation is done by the short method described above. 27 is the divisor and 42 is the quotient.

$$\begin{array}{r|l} 1134 & 27 \\ 5 & - \\ \hline 00 & 42 \end{array}$$

Excess of Nines in Division

The proof of the correctness of a multiplication by excess of nines has been explained. The same can be applied to division. The excess of nines in the divisor multiplied by the excess of nines in the quotient, and the excess of nines in this product added to the excess of nines in the remainder will give the excess of nines in the dividend if the operation has been done correctly. Take the division of 9763 by 281; the quotient is 34 and the remainder is 209. The excess of nines in the divisor, 281 is 2; the excess of nines in the quotient, 34, is 7; the product of these, 2×7 is 14; the excess of nines in this product is 5; the excess of nines in the remainder, 209, is 2; the sum of 5 and 2 is 7, and we find the proof that the division is in all probability correct in the excess of nines in the dividend, which is also 7. While it is quite conceivable that the excess of nines would come out right when the division was wrong, it is exceedingly improbable; if they do come out wrong the division is certainly incorrect.

If the process be compared with the similar operation for the proof of multiplication, it will be seen that the one follows from the other.

The example is given here.

$$\begin{array}{r}
 281 \overline{) 9763} \text{ (34} \\
 \underline{843} \\
 1333 \\
 \underline{1124} \\
 209
 \end{array}$$

Excess of 9's in the divisor, 281 is 2

Excess of 9's in the quotient, 34, is 7; $7 \times 2 = 14$;

excess of 9's is.....5

Excess of 9's in the remainder, 209 is 2, to be added

to above 2

—

7

Excess of 9's in the dividend, 9763 is..... 7

It follows that the division has been correctly performed because of the agreement in excess of nines.

Remainders in Division

When a number is used as a divisor of numbers indivisible by it without a remainder, the possible remainders are one less in number than itself.

Thus for 3, used as a divisor, when it divides a quantity giving a remainder, there can only be two possible remainders, 1 and 2. For the number four, used as a divisor, the only possible remainders are 1, 2 and 3. The same rule applies to all other divisors.

When the division is carried out beyond the decimal point, the remainder may give a continued or a complete decimal.

If divisions of numbers, indivisible by the divisors used, are carried out so as to give decimals in the quotient, interesting results are obtained.

With 2 as a divisor, the only possible remainder is .5.

With 3 as a divisor, there are two possible remainders, both continued decimals, .333 . . . and .666

With 4 as a divisor, the remainders are .25, .5, and .75.

With 5 as divisor, the remainders are .2, .4, .6 and .8.

With 6 as a divisor, the remainder may be .5, or continued decimals ending in 3's or in 6's.

With 8 as a divisor, the remainders end always in 5, and are one, two or three figure decimals, .5, .25, .125, .375, .625 and .875.

This leaves the numbers 7 and 9 to be considered.

The remainders given by the number 7 are repetends, identical in all cases, except that the decimals begin with different numbers; the order of the component digits is the same. The decimal remainders for the six possible remainders of seven used as a divisor, are the following:

.142857....

.571428....

.285714....

.714285....

.428571....

.857142....

The first of these remainders is divisible by 7 with a remainder of 1, reducing to the same decimal; this is why it is a repetend. The quotient obtained by dividing it by 7 is .020408+, a rather curious succession of numbers.

The number 9 gives as remainders continued decimals, of which each one is simply the repetition indefinitely of

the remainder figure. Suppose we have to divide 253 by 9. The quotient will be 28 and 1 over, or a remainder of 1. To put this into decimals we simply repeat the remainder indefinitely, as a continued decimal, and the quotient is 28.111

Suppose now that 290 is to be divided by 9; this gives a quotient of 32 and a remainder of 2; in decimals the remainder is 2 repeated indefinitely, as a continued decimal, and the quotient is 32.2222

This covers single digit divisors. The reader can try other divisors *ad libitum*, and will obtain quite interesting results.

Divisibility of Numbers

A number divisible by 2 is called an even number; such numbers end in one of the even numbers—2, 4, 6, 8 or else in a cipher, 0.

A number is divisible by 3 when the sum of its digits is divisible by 3. Thus the number 123 has as digits $1 + 2 + 3 = 6$. As this sum, 6, is divisible by 3 it follows that 123 is divisible by 3. The digits of 252 add up to 9; $2 + 5 + 2 = 9$. As 9 is divisible by 3 it follows that 252 is also. The quotients of the two examples given are 41 and 84.

Any even number divisible by 3 is divisible by 6. Thus the number 252 is an even number because it ends in 2; it is divisible by 3; therefore it is divisible by 6; divided by 6 the quotient is 42.

A number is divisible by 4 when its last two digits are divisible by 4. The last two digits of 9824, namely 24, are divisible by 4, therefore the whole number is; on trying it the quotient is found to be 2456.

All numbers ending in 5 are divisible by it.

All numbers ending in 0 are divisible by 5.

All numbers ending in 25 are divisible by it.

A number the sum of whose even placed digits is equal to the sum of the odd ones is divisible by 11.

Thus the sum of the even digits of the number 1538779 is 20, and the sum of the odd digits is also 20; the number therefore is divisible by 11.

If the last three figures of a number are divisible by 8 then the whole number is divisible by it. This is because 1000 is divisible by 8, so whatever precedes the last three figures it eventually comes to adding one or more thousands of them, which of course does not affect their divisibility by 8. Take 128; this is divisible by 8; if we add 1000 we have 1128, which of course if divided by 8 gives 141. 128 divided by 8 gives 16 and 1000 divided by 8 adds 125 to it making $16 + 125 = 141$. No matter how many figures are put before any three figures divisible by eight it only amounts to putting so many 1000's before the three, and each 1000 is divisible by 8. This property is practical as well as useful, and is a parallel case to the similar property of 4.

792 is divisible by 8. Put any numbers before it—say 33. This gives 33792. Dividing we find that 8 goes into 33 4 times and 1 over, so now we have to divide our original number increased by 1000, for that is what the effect of carrying 1 is, it gives 1792 as the number to be divided. It divides by 8 without remainder.

When the difference between the sum of the odd and even placed digits of a number is divisible by 11, the whole number is also divisible by it.

Take the number 54912; its even digits are 1 and 4,

whose sum is 5; its odd digits are 5, 9, and 2, and their sum is 16; the difference between 5 and 16 is 11; therefore the whole number is divisible by 11; the quotient is 4992. Take the number 27192; here the even digits have the larger sum, but the same rule holds; the difference is divisible by 11 and the whole number is so also.

If a number is indivisible by 4 without a remainder, and we divide it by 4 and carry it out in decimals, the quotients will contain either the decimal .5, the decimal .25, or the decimal .75 and no other.

Thus $22 \div 4 = 5.5$; $25 \div 4 = 6.25$; $27 \div 4 = 6.75$.

The above reduces in vulgar fractions to $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$. An analogous rule obtains for all such divisions; the remainders for division by three will be $\frac{1}{3}$ and $\frac{2}{3}$; for division by five the remainders will be $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$. The numerators of the remainders, when expressed as vulgar fractions will begin with one and go to one less than the divisor, and the divisor will be the denominator of the fractions.

The general statement of the above follows, together with its application to the single digits used as divisors of numbers giving remainders.

If the digits of any number are added together, their sum or the sum of the digits of their sum is the remainder which will be left on dividing the original number by 9.

Thus the sum of the digits of 875 is 20; the sum of the digits of 20 is 2; if we divide 875 by 9 the remainder will be 2, the sum of the digits of 20.

The sum of the digits of 962176 is 31; the sum of the digits of 31 is 4; if we divide the original number, 962176 by 9, the quotient will be 106908 with a remainder of 4, the sum of the digits of 31.

A number divisible by 2, 3, 4, 5, and 6 always with a remainder of 1, is divisible by 7 without any remainder.

The smallest of such numbers is 301. If it be divided by any of the five numbers cited, there will be a remainder of 1; if we divide it by 7, there will be no remainder.

Starting with 301 as a base, by successive additions of 420, other numbers of the same property can be obtained; such are 721, 1141, and others.

If the number 2519 is divided by any single number, it will give a remainder one less than the divisor. If divided by 2 the remainder will be 1; if divided by 3, the remainder will be 2 and so for the nine digits. It is the smallest number possessing this property.

Special Cases of Division

To divide a number by 5 multiply by 2 and cut off the last figure by a decimal point.

To divide 28 by 5, double it giving 56, and put the decimal point to the left of the last figure, 5.6 or $5\frac{6}{10}$. Or to take a larger number—714. To divide by 5 double it and place the decimal point to the left of the last figure, and we obtain 142.8.

To divide by 25 quadruple the dividend and cut off two figures.

Divide 1297 by 25. The product of 1297 multiplied by 4 is 5188, and placing the decimal point as directed we have as the quotient, 51.88.

This rule with a variation in the multiplier obtains for all powers of 5. For 125 as a divisor multiply by 8 and cut off three figures by the decimal point.

To divide any multiple of 11 by 11 proceed as follows:
Put down the right-hand digit of the dividend as the

right-hand figure of the quotient. Subtract this from the next figure of the dividend, carrying 1 if necessary. The result of the subtraction gives the next figure of the quotient. The process is carried out to the end.

Divide 56408 by 11. Put down 8 as the right-hand figure of the quotient. Subtract 8 from the next figure, 0, of the dividend; this gives 2, the second figure of the quotient, with one to carry. Adding the 1 to be carried to the second figure of the quotient gives 3, and 3 subtracted from the next figure of the dividend, 4, gives 1, the third figure of the quotient, with nothing to carry. Finally this third figure of the quotient is subtracted from the next figure of the dividend, 6 giving 5, the fourth figure of the quotient, leaving 5 of the quotient to be subtracted from the last figure of the dividend, 5, leaving as the quotient, 5128, these being the figures found.

Dividing by Ninety-Nine

To divide by 99, add the two right hand figures of the dividend to the rest of the number and put it down under the original number. Do the same with the smaller number, and put it down under the other two. Keep up this process until 99 is left or a quantity less than 99. Cut off the two right hand figures of everyone of the quantities, either mentally or by drawing a vertical line; add up what is left on the left of the line; this is the quotient. The number at the bottom of the right hand of the line is the remainder; of course if the remainder is 99, 1 has to be added to the quotient.

Divide 869432 by 99.

The two right hand figures are 32; these are added to the remaining figures, thus: $8694 + 32 = 8726$. This

is the first quantity to be added to the original dividend. To get the next number we cut off from 8726 its two right hand numbers, 26 and add them to what is left of the number; we have thus: $87 + 26 = 113$. This is the second number to be added to our original dividend. The same operation is applied to 113; its two left hand digits are cut off and added to what is left; this gives $1 + 13 = 14$. As this is only a two digit quantity nothing more is to be done to it. We now set down the quantities obtained and draw the vertical line as below.

$$\begin{array}{r}
 8694 \overline{) 32} \\
 \underline{87} \\
 1 \\
 \underline{14} \\
 8782 \text{ with a remainder, } 14.
 \end{array}$$

This is the result of dividing 869432 by 99.

Divide 23661 by 99.

Proceeding exactly as before we have as our quantities 23661, $236 + 61 = 297$ and $2 + 97 = 99$; these we treat as before:

$$\begin{array}{r}
 236 \overline{) 61} \\
 \underline{297} \\
 99
 \end{array}$$

238 with a remainder 99, giving 239 as the quotient.

Properties of the Number Three in Division

If we take any two numbers we will find that either their sum or their difference or one or both of the numbers is divisible by three. 19 and 17 seem rather hopeless, but their sum is 36, and 36 is divisible by 3.

Any number the sum of whose digits is divisible by 3 can be divided by 3 itself. A clue to the reason for this lies in the fact that if we multiply all the single digits by three we shall find the nine digits represented in the terminal figures. Three times runs—3, 6, 9, 12, 15, 18, 21, 24, 27—these are the products of 3 by the nine single numbers and the same nine numbers are there in the last places. Now take a number, say 74228115—the sum of its digits is 30, this is divisible by 3. Now try the number itself and its quotient when divided by 3 is 24742705 with no remainder. The numbers 7 and 9 also have the nine digits as the last figures in their multiplications by the nine first figures, but that does not confer this property on 7, but 9 possesses it.

Suppose a number is not divisible by 3, then the amount by which the sum of its digits exceeds the nearest lower multiple of 3 will be a number which if subtracted from the original one will leave it divisible by 3 without a remainder. Take the number 3983; the sum of its digits is 23; the next lower multiple of 3 is 21, and 2 therefore is the difference to be subtracted from the original number; carrying out the subtraction we get 3981, which is divisible by 3, the quotient being 1327 and there is no remainder.

Lewis Carroll's Short Cut

This is Lewis Carroll's short cut for dividing any multiple of 9 by 9. It is only a matter of interest, and in great measure such, because of its distinguished originator, the author of "Alice in Wonderland" and of "Alice in the Looking Glass"; the originator of the Wonderland arithmetical rules of Ambition, Distraction, Uglifi-

cation and Derision. (Alice in Wonderland, Chap. IX.)

Write down the number to be divided; put a cipher over the units figure and subtract the units figure from the cipher. The result is the units figure of the quotient. Then put this figure over the tens figure of the number to be divided and subtract the tens figure of the original number from it and put it down as the tens figure of the quotient. Write it also above the hundreds figure of the number and subtract the hundreds figure from it and put it down as the hundreds figure of the quotient and also put it over the thousands figure and so on until the number runs out.

Divide 36459 by 9.

Write out the dividend, 36459. Put a cipher over the right hand or units figure, subtract and put the figure down as the units figure of the quotient and also put it over the second or tens figure of the number, in this case, 5. The process is repeated according to the directions. The successive steps are given here:

10	510	0510	40510
36459	36459	36459	36459
51	051	4051	4051 the quotient.

Inspection of the last operation shows that it all is simply subtracting the dividend from ten times the quotient.

CHAPTER VI

FRACTIONS

Vulgar Fractions

Any quantity numerically less than unity is a fraction. A vulgar fraction is one expressed as a division; the divisor classifies the fraction, as being of the half class, the thirds class, the thirtieths class or any other class whatever. A definite one of these divisors, which are called denominators, is taken for each fraction—the figure expressing the dividend is called the numerator.

To write a fraction the numerator is placed above a short horizontal or diagonal line or bar, and the denominator is placed below the same bar.

$\frac{1}{3}$, $\frac{12}{30}$ are fractions; the first belongs to the class of thirds, as its denominator tells us, the next one belongs to the class of thirtieths as its denominator indicates. The numerator of the first fraction tells that only one of its class is taken, the numerator of the next one tells that twelve of its class are taken.

Meaning of the Fractional Bar

One most important thing to be realized about fractions is the true significance of the bar. It is a sign of division, just as truly as is the sign $\div$. $\frac{5}{13}$ may be expressed in words as five thirteenthths or as five divided by thirteen; it could correctly be written $5 \div 13$.

The fractional bar may be and often is used as a sign of division. $125 \div 25 = 5$ can be written $\frac{125}{25} = 5$; both mean identically the same thing.

The fractional bar is sometimes an oblique one; this is merely a matter of taste or of convenience.

Although it is never used as the sign of division, a fraction might be written perfectly correctly as the indication or setting down of a calculation in division; $\frac{3}{16}$ might be written $16)3$, for this is the carrying out to its full meaning of the fraction with its bar. If the division is completed the result may be expressed in decimal fraction.

Changing the Value of a Fraction

There are two ways of increasing the value of a fraction; one is to increase its numerator, the other to diminish its denominator. The reverse also is true, if the denominator is increased or the numerator diminished the value of the fraction will be diminished.

Thus $\frac{3}{7}$ is smaller than $\frac{4}{7}$ and also smaller than $\frac{3}{6}$; it is larger than $\frac{3}{8}$ or than $\frac{2}{7}$. The reader will observe how the numerators and denominators are increased and diminished in these examples.

Reducing Fractions to the Same Class or to a Common Denominator

If you want to add a gallon to a pint you must reduce them to the same class either to gallon or to pint class; the sum of the two is one and one eighth gallon or is nine points. To add two fractions they must be of the same class, that is to say must have the same denominator.

Addition and Subtraction of Fractions

To add $\frac{1}{2}$ to $\frac{1}{3}$ we must first multiply numerator and denominator of each fraction by the denominator of the

other; this gives $\frac{3}{6}$ and $\frac{2}{6}$, and the sum of these is $\frac{5}{6}$; the numerators are added because they tell how many there are of each—if there are two of them to be added to three of them the result is, of course, 5. The things added are sixths, and there are five of them.

For subtraction the reverse is carried out; the reduction to a common denominator is effected and one numerator is subtracted from the other one. $\frac{1}{3}$ subtracted from $\frac{1}{2}$ would give $\frac{1}{6}$.

The addition of fractions of different denominators can be expressed in words without reduction to a common denominator; the above addition can be called a half and a third. This is constantly done in adding fractions to whole numbers—two added to a half is usually expressed as two and a half. It might be expressed as $\frac{5}{2}$ or five halves.

Multiplication and Division of Fractions

If we multiply a number by a fraction whose numerator is 1, we can equally well divide the same number by the denominator. $2 \times \frac{1}{2}$ or $2 \div 2$ give the same result, namely, $\frac{2}{2}$ or 1.

Any number may be regarded as a fraction; as the denominator of a fraction expresses what may be called its class, and as a whole number refers to the class of units, if a whole number is written as a fraction its denominator must be 1. Thus any whole number can be written over the fractional bar as a numerator, with the denominator, 1, under the bar. 125 can be written $\frac{125}{1}$ for instance. As nothing is gained by doing this it is never done, just as the decimal point is omitted in writing whole numbers. As a matter of correctness the number

125 should have the decimal point expressed, 125., although it is always omitted, unless there is a decimal fraction to go there.

In the same way the bar and unit denominator are always omitted in writing whole numbers; nothing would be gained by putting them down.

As the fractional form indicates division, if we are to divide a number by another it can be done directly by division, or the divisor can be made the denominator of a fraction whose numerator is 1, and the dividend can be multiplied by the fraction.

Dividing by a number, or multiplying by a fraction having the number in question as a denominator and having 1 for its numerator, is identical.

If 25 is to be divided by 5 the operation may be effected by ordinary division giving 5, or it may be expressed as $25 \times \frac{1}{5}$.

To multiply by a fraction multiply by its numerator and divide by its denominator.

Thus $25 \times \frac{1}{5}$ is 25 multiplied by 1 and divided by 5, which gives $25 \div 5$ which is 5.

To multiply a fraction by a whole number its numerator may be multiplied by the number. To multiply $\frac{1}{5}$ by 25 gives, if we allow this rule, $25 \div 5$ exactly as above, as it should be.

Taking a fraction with a numerator greater than unity, say $\frac{2}{4}$ to multiply it by the same fraction as in the previous example, namely $\frac{1}{2}$, we can equally well divide its numerator by the denominator of the multiplier or by 2; this division can be done in another way; the denominator of the multiplicand, $\frac{2}{4}$, can be multiplied by 2; the first operation gives us $\frac{1}{4}$, the other operation gives $\frac{2}{8}$; both these are of identical value.

The general rule for multiplication by a fraction having a numerator 1, is to multiply the denominator of the multiplicand by the denominator of the fractional multiplier.

Now suppose the fraction multiplier had a number greater than unity for its numerator; it is clear that it would be that many times larger, so to multiply by it, we should multiply the numerator of the multiplicand by the numerator of the fractional multiplier, or, what is the same thing, we should divide the denominator of the multiplicand by it. Next we should divide the numerator of the multiplicand by the denominator of the multiplier. The result is the same, but it is simpler to multiply the denominators and numerators together. Therefore to multiply two fractions, multiply the numerators together for a new numerator and the denominators for a new denominator.

A fraction can also be multiplied by a whole number by dividing its denominator by the multiplier. To multiply $\frac{1}{5}$ by 25 by this method, we would obtain the expression,

$\frac{1}{5 \div 25}$ which is equal to $1 \div \frac{1}{5}$ which is 5. This is the same result as that obtained by the other methods.

If a number is to be multiplied it means that the number is to be taken as many times as there are units in the multiplier. If the multiplier is a fraction, say $\frac{1}{2}$ it has one half a unit in it; if a number is multiplied by $\frac{1}{2}$ therefore it can be taken one half time; $10 \times \frac{1}{2} = 5$.

If a fraction is multiplied by another fraction the same principle applies. $\frac{1}{2} \times \frac{1}{2}$ means that $\frac{1}{2}$ is to be taken one half time, which gives of course $\frac{1}{4}$. If $\frac{1}{2}$ is to be multiplied by $\frac{3}{4}$ it must first be taken three times, because of the numerator, 3, and then one fourth time because of

the denominator, 4. If we take $\frac{1}{2}$ three times it gives $\frac{3}{2}$; if next we take $\frac{3}{2}$ one fourth time we have $\frac{3}{8}$, which is one fourth of $\frac{3}{2}$.

If the same operation of division or multiplication is effected on both numerator and denominator of a fraction, it is evident that the ratio of numerator to denominator will be unchanged.

$\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ are all the same in value; the last two are obtained by multiplying numerator and denominator by 2 and by 3; the ratios are the same in all.

If we divide the denominator of a fraction by one tenth of itself and the numerator by the same the value of the fraction will be unchanged but we will obtain a fraction whose denominator will be a multiple of ten.

This is a clumsy way of reaching the desired result; it is easier to divide the numerator by the denominator with due regard to the decimal point. But as a matter of explaining it is well to try it the first way.

Conversion of Vulgar Fractions into Decimal Fractions

Suppose we divide the two elements of the fraction, $\frac{1}{2}$ by one tenth of the denominator. We have for the numerator $1 \div .2 = 5$, and for the denominator, $2 \div .2 = 10$, and the new fraction is $\frac{5}{10}$. The value is unchanged and as it is a decimal in its denominator, it can be written as a decimal fraction, .5.

The more direct way, and the way always used to get a decimal fraction from a vulgar fraction, is the direct division of the numerator by the denominator.

To reduce $\frac{1}{2}$ to a decimal we divide as follows:
2) 1.0; the same result as obtained above in the more

.5
indirect way.

Every decimal has a denominator understood; it is 1 followed by as many ciphers as there are figures in the decimal, ciphers coming between the characteristic figure or figures and the decimal point counting as figures.

If the characteristic figures of a decimal are taken as whole numbers and are placed over the fractional bar with the denominator determined as above below the bar the decimal will be expressed as a vulgar fraction.

Thus .5 is the same in value and can be expressed as $\frac{5}{10}$; .05 as $\frac{5}{100}$, and so on.

This is one way of expressing a decimal as a vulgar fraction and it gives the way in which it is always expressed in words; .5 is called five tenths, .05 is called five hundredths.

Another way to reduce a decimal is different in its result; it gives a fraction of the same value, but one not having ten or a multiple of ten for a numerator. It is the reverse of the operation just described. It consists in expressing the decimal as a vulgar fraction, and then dividing the characteristic figure or figures of both numerator and denominator by the characteristic figures of the numerator.

.5 is expressible as $\frac{5}{10}$; dividing both numerator and denominator by 5 gives $\frac{1}{2}$, which is the value of the decimal.

We may also divide both parts of the fraction by any common divisor, that is to say by any number which will divide both parts without remainders. Take the decimal .75; this can be written $\frac{75}{100}$; both the numerator and denominator are divisible by 25; doing this gives the vulgar fraction, $\frac{3}{4}$, which is the value of .75. Or both might have been divided by 5 giving $\frac{15}{20}$, also of the same value as .75.

CHAPTER VII

THE DECIMAL POINT

Errors Due to Misplaced Decimal Point

No class of errors in arithmetic are so frequent as those due to the decimal point. Put in the wrong place the smallest error it can introduce is a multiplication or a division by ten, and the error may be larger to any extent whatever, but can never be less. There is a tendency to trust to one's understanding in the matter of the decimal point, and the everyday expression of percentage as an integral figure, instead of a decimal, is an example of a source of confusion.

Six per cent. is written 6% ; it is fair to say that many never picture 6% as really indicating a multiple which is a decimal, .06 or a vulgar fraction, $\frac{6}{100}$ ths.

Any percentage for purposes of calculation is completely and properly expressed as a decimal. Five per cent. of one hundred dollars is properly obtained by multiplying \$100 by .05, giving \$5.00 as the answer.

Addition of Decimals

In adding decimals the same rule is followed as in adding integral numbers. The first rule of addition is to put units under units and tens under tens and so on. On the right of the decimal point in putting down an addition of decimals and in carrying out the operation, put tenths under tenths and hundredths under hundredths and so to the last of the decimals in the numbers.

Follow the same rule you follow when adding up dollars and cents. A cent is one hundredth of a dollar and is written correctly \$.01; ten cents are the tenth of a dollar, \$.1, which however, as we never think of this amount as a dime, but always as ten cents, is written \$.10. The final 0 does not affect its value in any way whatever.

Now suppose five dollars and fifteen cents are to be added to four dollars and twenty five cents.

The addition is carried out with dollars under dollars and cents under cents.

$$\begin{array}{r} \$5.15 \\ 4.25 \\ \hline \$9.40 \end{array}$$

Suppose five and fifteen hundredths gallons of turpentine were to be mixed with four and twenty five hundredths gallons of linseed oil and that there was no shrinkage what would the total amount of the mixture be? The numbers would be put down precisely as above except that the dollar mark would be omitted. Units would go under units, which brings the 4 under the 5; then going to the right of the decimal point we put the 2 under the 1, because they are both tenths; 5 is put under 5 because they are both hundredths. If an error is made in putting the digits where they belong the result will be incorrect.

Although the cipher is always used in writing such amounts as ten cents or fifty cents, the latter can just as correctly be written \$.1 and \$.5 as \$.10 or \$.50. The sum of the last example could have been written \$9.4.

Ciphers never need be placed after the last digit of a decimal.

Suppose in some complicated calculation the above amount had had its decimal point misplaced, say one place wrong to the right—it would then have read \$94.

This is an example of a frequent error in arithmetical work.

By following the rule of keeping units under units and tenths under tenths and so for all integers and for all decimals no misplacing of the decimal point should occur in addition.

No one would ever think of adding dollars to cents in the same column; precisely the same applies to decimals when added.

Subtraction of Decimals

The same applies to subtraction of decimals. To subtract one of the numbers of the last example from the other, they would be written down in exactly the same way; hundredths would have been subtracted from hundredths, tenths from tenths and units from units and the result or remainder would have been, .90 or .9, for both of these are the one and same; it is only a matter of convenience which way you select to write them.

Here a difference appears between decimals and integers in practice; as many ciphers as are convenient may be placed to the right of a decimal without changing its value and for one reason or another ciphers are often so placed, but it is not so often that ciphers are placed to the left of an integral number, although it is perfectly logical to do so.

Multiplication of Decimals

It seems a matter of carelessness to make an error in the decimal point in addition or subtraction; in multi-

plication and division it is not an impossibility and it is not infrequent.

To multiply decimals or mixed numbers there is no necessity of putting units under units and tenths under tenths, although as a matter of discipline and good practice it is to be recommended.

The decimals in a product are equal to the sum of the decimals in the numbers multiplied. If in the multiplication a final 0 appears in the product, it counts as a decimal and should be written because it has to be taken into account in fixing the place of the decimal point.

Multiply 9.5 by 6.3, and also 7.5 by 5.2.

9.5	7.5
6.3	5.2
285	150
570	375
59.85	39.00

Both multiplications are done regularly as in the case of integral numbers; each of the original numbers has one decimal, therefore their product must have two decimals. In the case of the first both are significant figures, and have a value in addition to fixing the place of the decimal point; in the case of the second product there are two ciphers; these are without numerical value and can just as well be omitted, but have to be put down to fix the place of the decimal point.

Placing the Decimal Point

A decimal may be defined as a multiple of ten or as some power of ten or as a dividend of some power of ten. If the latter it is called a decimal fraction.

The numbers, 190, 100 and the like are decimal numbers; the fractions $.125$, $.076$ and the like are decimal fractions.

It would be perfectly correct to write these decimal fractions, $\frac{125}{1000}$, $\frac{76}{1000}$; they would then be expressed as vulgar fractions and would be so termed, although strictly speaking they are decimals.

When an integral number is written it is customary to omit the decimal point. It is understood and is taken as being placed just to the right of the unit figure or cipher in the unit place of the number.

To multiply a whole number by 10, on account of the decimal point being left to be understood, all that is necessary is to place a cipher on the right. Thus $125 \times 10 = 1250$. If we had written the decimal point to the right of the original number, 125., which would have been perfectly correct, then to multiply by 10 we should have had to rub out the point, to put down the cipher directly after the 5, and if we wished we could place a new decimal point next to the 0, thus 1250., although it would be usually quite unnecessary.

But in the case of decimal fractions the decimal point can never be omitted.

To multiply a decimal fraction by 10 the decimal point is moved one place to the right; to multiply by 100 it is moved two places to the right. Thus $.125 \times 10 = 1.25$, which operation may be expressed in vulgar fractions thus: $\frac{1}{8} \times 10 = 1\frac{2}{8}$ or $1\frac{1}{4}$. In general any multiplication by a decimal multiplier is done by moving the decimal point as many places to the right as there are ciphers in the multiplier.

In the case of whole numbers the multiplication effected

by the annexing of ciphers on the right hand of the number, amounts to a shifting of the decimal point to the right; this would have to be done were it not customary to omit the decimal point in putting down whole numbers.

Division of Decimals

To divide a decimal fraction by a decimal number the reverse operation has to be done; the decimal point is moved one point to the left for every cipher in the divisor. Thus to divide .125 by 10 it is written .0125; to divide it by 100 write it .00125.

In the division of whole numbers by decimal divisors the same rule is followed. To divide 125, a whole number, by 10 the unexpressed decimal point is moved one place to the left, and we have $125 \div 10 = 12.5$. If it was to be divided by 100 it would become 1.25.

A number may lie upon both sides of the decimal point; it is then a mixed number. Such are 12.5 and 1.25, with whole numbers on the left of the decimal point and decimal fractions on the right. They may be written as improper fractions, $\frac{125}{10}$ and $\frac{125}{100}$, or as mixed numbers, $12\frac{5}{10}$ and $1\frac{25}{100}$, which reduce to $12\frac{1}{2}$ and $1\frac{1}{4}$.

In division of one number by another there is no need that the divisor shall be smaller than the dividend. The decimal point takes care of the relations of one to the other and enables the division to be carried out as far as desired if a division without remainder is impossible. A vulgar fraction is merely a statement of division to be done, the divisor being the denominator and usually larger than the numerator, which is the dividend.

The fraction, $\frac{1}{2}$, is merely the expression of a division to be done, the division of 1 by 2; the fraction $\frac{1}{25}$ is the expression of the division of 1 by 25.

If such divisions are carried out we obtain decimal fractions and here is where the decimal point must be watched. Dividing 1 by 2 gives .5 or five tenths; dividing 1 by 25 gives .04 or four one hundredths.

A mixed number is treated in exactly the same way. Take $12\frac{3}{5}$. Dividing 3 by 5 gives .6; this is annexed to the whole number and we have 12.6 or $12\frac{6}{10}$. Or we may write $12\frac{3}{5}$ as an improper fraction and carry out the division; $12\frac{3}{5} = 6\frac{3}{5} = 12.6$. All is done exactly as in the division of whole numbers but with close regard to the decimal point.

The insertion of a cipher between a decimal fraction or a whole number and the decimal point, or the removal of a cipher therefrom does affect its value as just described.

It is in the division of and by decimal fractions that the greatest liability to error occurs.

The rule is that the decimals in the quotient are equal to those in the dividend diminished by those in the divisor. If the divisor has more figures to the right of its decimal point than the dividend has, ciphers are to be annexed to the dividend to make the decimals equal in number.

Divide 1.25 by .25. Express it as a regular division and go through the operation. $.25)1.25(5$. As the decimals in both are equal there are none to go into the quotient.

Divide .125 by .25. We have $.25).125(.5$. Here there is one more decimal in the dividend than in the divisor, therefore there is one decimal in the quotient.

Divide 125 by .25. We annex two ciphers to the dividend so that it shall have as many decimals as the divisor; this gives: $.25)125.00(500$. As the decimals in the divisor are as many as those in the dividend there are no decimals to go into the quotient.

Although the annexing of the ciphers could have been dispensed with, and the decimal point could have been determined mentally, for the sake of certainty it is best to add ciphers until there are at least as many in the dividend as in the divisor. It is immaterial if there are more.

Divide .001 by 95. Add enough ciphers to make the dividend numerically equal to or greater than the divisor, irrespective of the decimals. Thus we have:

$$95).00100(.00001;$$

or carrying it still further:

$$95).0010000(.0000105.$$

In each case, as there are no decimals in the divisor, all that is necessary is to make the decimals in the quotient equal in number to those in the dividend.

CHAPTER VIII

INTEREST AND DISCOUNT. PERCENTAGE CALCULATIONS

Expression of Rate of Interest

The rate of interest is always expressed as an integral or mixed number, as five per cent, six per cent, six and a half per cent—5 p. c., 6 p. c., $6\frac{1}{2}$ p. c.—as the case may be.

The correct way to write it is as a decimal, and perform the calculations into which it enters, with due regard to the decimal point. Thus six per cent is correctly written as .06—five per cent as .05 and so on.

Calculating Interest

To calculate 5 p. c. interest on \$762.98 multiply the principal by .05.

$$\begin{array}{r} 762.98 \\ .05 \\ \hline 38.1490 \end{array}$$

$$\begin{array}{r} 7.6298 \\ 5 \\ \hline 38.1490 \end{array}$$

$$\begin{array}{r} 2 \) \ 76.2980 \\ \hline 38.1490 \end{array}$$

As there are four decimals in the two numbers there must be four in their product. In the first example the operation is carried out as a regular multiplication. In the second the decimal point is shifted two places to the left and the multiplier is taken as the figure indicating the percentage rate. Both operations are the same in essentials; another way to do the calculation is to divide

by 2, as in the third example, after shifting the decimal point.

The principal as regards its integral figures was stated in dollars; the answer reads therefore \$38.15.

In interest calculations the decimal point should be kept in mind and used correctly—guessing at how many dollars and how many cents are in the result of a multiplication is frequently done; if it be remembered that a per cent is properly written as a decimal and if the placing of the decimal point is perfectly understood, the first method is the best way to work.

Since a per cent expresses really a stated number of hundredths it can be written as a vulgar fraction; six per cent can be written $\frac{6}{100}$. Interest calculations can be done with this expression.

Take \$97.63 as the principal and calculate interest at 7 p. c., using vulgar fractions

$$\begin{array}{r}
 97.63 \\
 7 \\
 \hline
 100 \) \ 683.41 \\
 \hline
 \$6.83
 \end{array}$$

Short Ways of Calculating Interest

If the 360-day year and the 30-day month is used in interest calculations, the process for 6% is simplicity itself. The interest for a year is 6%; the interest for a month is $\frac{1}{2}\%$; the interest for a day is $\frac{1}{30}$ th of that for a month. With this as a starting point other standard rates of interest are calculated nearly as simply.

If the 365-day year is used then each interest may as well be calculated in the regular way.

Assuming that we do calculate the interest on any sum at the rate of 6%, we can easily derive the others from it.

For 3% divide the 6% amount by 2.

For 4% subtract $\frac{1}{3}$ from the 6% amount.

For $4\frac{1}{2}\%$ subtract $\frac{1}{4}$.

For 5% subtract $\frac{1}{6}$.

For $2\frac{1}{2}\%$ divide the last by 2.

For 7% add $\frac{1}{6}$.

For $7\frac{1}{2}\%$ add $\frac{1}{4}$.

For 8% add $\frac{1}{3}$.

For 9% add $\frac{1}{2}$.

There are other ways of doing interest calculations in a short way. It will be sufficient to give the methods, as the carrying of them out presents no difficulties.

For 4% divide the principal by 25. This is not a short way; in any case it is only an alternative method, and explains what 4% means.

For 5% divide the principal by 20.

For 2% divide the principal by 50.

Calculate the interest on \$379.68 at 4% and at $4\frac{1}{2}\%$ for three months and 10 days, taking the year at 360 days.

One month's interest at 6% is the half of 1%, and this is \$1.89; for three months it is \$5.67; ten days is taken as the third of a month so the interest at 6% for that period is \$0.63, and the sum of the two last is \$6.30. For 4% subtract one third or $6.30 \div 3$; this leaves \$4.20; for $4\frac{1}{2}\%$ subtract one quarter; this gives \$4.73. The answers are therefore \$4.20 and \$4.73.

Reduction of Interest Periods

One half of 1%, which is in decimals .005, is one month's interest at 6% on the 360 day basis for one month.

This is reduced to other rates by the methods just given. It can be reduced to other periods of time by addition, multiplication or division. One per cent is the interest for two months at 6%. The first column gives the reduction of one month to other periods; the second column the reduction for two months.

To reduce one month to				To reduce two months to			
120	days	multiply	by 4	120	days	multiply	by 2
90	"	"	" 3	90	"	"	" $1\frac{1}{2}$
45	"	"	" $1\frac{1}{2}$	45	"	"	" $\frac{3}{4}$
20	"	"	" $\frac{2}{3}$	20	"	divide	" 3
15	"	divide	" 2	15	"	"	" 6

One Day's Interest

To calculate the interest for one day on any amount, the product of the amount by the interest is divided by the number of days in the year. If we assume the year to consist of 360 days the quotient obtained by dividing 360 by the interest rate divided by 100 will give a divisor by which if the principal be divided the quotient will be the interest on the amount in question for one day.

Suppose the interest at 6% is to be determined on \$100.00 for one day. To obtain the general divisor divide the interest rate, 6, by 100. This gives .06. Dividing 360 by .06 gives 6000, the general divisor for 6%. Dividing \$100.00 by 6000 gives .01666 or $1\frac{2}{3}$ cents as the required interest.

The reason for using the above general divisor is that it saves one operation, the multiplication by the interest rate. To multiply by .06 and divide by 360 is the same thing as to divide by 6000—the operations are identical.

The above method of calculating interest is applied to

any number of days by multiplying the one day's interest thus determined by the number of days. Thus for ten days the interest on the above at the given rate would be $16\frac{2}{3}$ cents—for thirty days it would be thirty times the amount for one day, or $1\frac{2}{3}$ cents multiplied by 30, which gives 50 cents.

Divisors for Rates of Interest

General divisors for other rates of interest follow. They are calculated only for such rates as give integral divisors. If a fraction is in the divisor the process is of little or no practical value.

All the divisors are based on the 360 day year.

To use the table multiply the principal by the number of days and divide the product by the general divisor for the specified interest rate.

$2\frac{1}{2}\%$	14400	6%	6000	12%	3000
3%	12000	8%	4500	15%	2400
4%	9000	9%	4000	16%	2250
$4\frac{1}{2}\%$	8000	10%	3600	18%	2000
5%	7200			20%	1800

What is the interest on \$1791.23 for 39 days at 4%?

Multiply the principal by the days— $1791.23 \times 39 = 69857.97$, and divide the product by the general divisor for 4%; $69857.97 \div 9000 = 7.762$ or \$7.76.

Cancellation in Interest Calculations

Cancellation is often applicable in interest calculations, especially when the 365 day year is the basis. On the left of the line place the days in the year; on the right of the line place the interest rate, the principal and

the number of days for which the interest is to run.

What is the interest on \$1575.25 at 4% for 27 days;
 $\frac{365}{360}$ day year?

$$\begin{array}{r|l}
 365 & 1575.25 \\
 73 & 315.04 \\
 & .04 \\
 & 27 \\
 \hline
 73) 340.2540 & (4.67
 \end{array}$$

The interest is \$4.67 on
 the 365 day year basis.

$$\begin{array}{r|l}
 360 & 1575.25 \\
 72 & 315.05 \\
 & .04 \\
 & 27 \\
 \hline
 72) 340.2540 & (4.73
 \end{array}$$

The interest is \$4.73 on
 the 360 day year basis.

The rate of interest paid for money, when bills due are not discounted, is not always realized by those failing to take advantage of discount for quick payment, or by those offering it. A usual system is to offer 1% or 2% discount if the bill is paid in ten days. Suppose the bill is paid in any case in thirty days; then for the extra twenty days the 1% represents a rate of about 18% per annum, and the 2% is a rate of 36% per annum. Sixty days net, less 2% ten days, is $14\frac{2}{5}\%$ per annum. Even if we take a most moderate case we will find that thirty days net, less $\frac{1}{2}\%$ for cash in ten days, is at the rate of 9% per annum.

Percentage Calculations

To calculate what per cent one number is of another divide the first number by the second, or divide the percentage number by the base.

What per cent is 99 of 108? Proceeding as above we find: $99 \div 108 = .9166$ or expressing it as a percentage, 91.66%.

What per cent of 99 is 108? Here we must divide the other way, for the percentage amount must always be divided by the base. Then $108 \div 99 = 1.0909$ or 109.09%.

Suppose a number is increased by a certain per cent. The question may be asked what per cent must be subtracted from the increased number to get the original number again. It will be a different per cent.

If 10% is added to 50 it will give 55. This is because 5 is 10% of 50. So to get 50 back again we must subtract 5 from our 55. But 5 is not 10% of 55. To find what per cent it is we must divide 5 by 55. $5 \div 55 = .0909$ which is 9.09%.

A city has 100,000 inhabitants. Another city is 50% larger; how much smaller is the first than the last city? The larger city has 150,000 inhabitants; this is because 50,000 is 50% of 100,000, but as 50,000 is $33\frac{1}{3}\%$ of 150,000 the smaller city is at one and the same time $33\frac{1}{3}\%$ smaller than the 150,000 inhabitant city.

If the above operations are followed out with regard to the decimal point, it will be seen that the percentage sign, %, or the word per cent, applied to an integral or mixed number, has the effect of dividing it by 100, because any stated percentage is as many hundredths of the base number as there are significant figures in the percentage. Thus 6 per cent or 6% or .06 all indicate the same thing.

Approximate Calculations by Percentages

By using percentage corrections or fractional ones approximate results, near enough correctness to be used in practice, may often be obtained. Examples are very

numerous; the point is to get the principle well understood and then it can be applied to any new case.

A kilometer is about .62 of a mile. To reduce kilometers to miles we should multiply by .62. This is a two figure multiplication. It is easier to multiply by .6 and add $\frac{1}{30}$ or even 3%.

Reduce 23 kilometers to miles by the three methods cited above.

$$a. 23 \times .62 = 14.26 \text{ miles.}$$

$$b. 23 \times .6 = 13.8$$

$$13.8 + .46 = 14.26 \text{ miles.}$$

$$c. 23 \times .6 = 13.8$$

$$13.8 + .42 = 14.22 \text{ miles.}$$

In example *b* $\frac{1}{30}$ is added; in example *c* 3% is added; for all ordinary purposes one method is as good as the other.

A mile is equal to about 1.6 kilometers. Suppose we had to reduce 23 miles to kilometers.

We may multiply 23 by 1.6. The easy way to do this is to multiply by 8, giving 184 and then to multiply this by 2, giving 368 kilometers.

Or else add to the miles .6 of their number; $23 \times .6$ gives 13.8, which added to 23 gives 36.8 kilometers as before.

A meter is equal to a little more than 39 inches or a little more than $3\frac{1}{4}$ feet. To reduce meters to feet multiply by 3 and add $\frac{1}{12}$.

1100 meters are equal approximately to $(1100 \times 3) + (3300 \times \frac{1}{12})$ which is 3575 feet.

It may be done by percentage; we may multiply by 3 and add 8%. The result will be nearly the same; for the 1100 meters we should find $3300 + 8\% = 3564$. This is not so close, the percentage should be $8\frac{1}{3}\%$.

The fraction method is preferable and just as easy as the percentage method.

A kilogram is equal to 2.2 lbs. avds. To reduce kilograms to pounds multiply by 2 and add 10%. Thus 25 kilograms are equal to 50 lbs. plus 5 lbs. or 55 lbs.

The 10% may be added before the multiplication— $25 + 10\% = 27\frac{1}{2}$ and $27\frac{1}{2} \times 2 = 55$ just as before.

A pound avds. is equal to about .454 kilogram. Therefore to reduce pounds to kilograms divide by 2 and subtract $\frac{1}{10}$. For 25 pounds the first method gives: $25 \div 2 = 12.5$; and $12.5 - 1.2 = 11.3$ lbs. The ten per cent of 12.5 is taken as 1.2.

The English stone, used as a unit of weight, especially for men, is equal to 14 lbs. To reduce stones to pounds multiply by one half the number of pounds in a stone, namely by 7, and then multiply by 2. Thus if a man weighs 13 stones, to reduce the stones to pounds, multiply by 7, which gives 91, and multiply this by 2, which gives the exact weight in pounds, 182 lbs.

This last reduction is not approximate, as are all the preceding ones in the last page.

Then to reduce pounds to stones divide by 7 and then by 2 or the other way. To reduce 199 lbs. to stones; dividing by 2 gives $99\frac{1}{2}$ and this divided by 7 gives $14\frac{2}{14}$ stones, which is the answer.

CHAPTER IX

POWERS OF NUMBERS

Powers and Roots

The square of any number above 3 has two or more digits: the square of any number above 9 has three or more digits; there is no rule for determining when a series of values, with one more digit in each case than in those of the preceding series begins.

Powers and Roots of Decimal and of Mixed Numbers

The square of a decimal fraction must have at least two decimal places, and the number of decimal places must be an even number.

Thus the square of .2 is .04; the square of .4 is .16; the square of .11 is .0121.

It follows from the above that if we have to extract the square root of a single figure decimal, such as .4 or .9, we must add a cipher and use the number with cipher annexed as the first period for the extraction.

Thus the square root of .4 has to be taken as the square root of .40, or of .4000, or of .400000, and so on, carrying it out to as many places as desired, always annexing two ciphers, in accordance with the rule for extraction of the square root. The square root of .4 is .632+; that of .9 is .948+; the fact that on their face they appear to be single digit squares has nothing to do with the value of their true square roots.

Analogous laws for higher powers could be given, but the above is sufficient to illustrate the laws affecting the powers of decimal numbers.

It will be seen also that the powers of decimals, when such powers are greater than the first power, and the same applies to fractions, are smaller in value than the original quantities. For corresponding negative powers the reverse is the case; the powers for decimals and fractions are larger than the original quantities, for whole numbers negative powers are smaller.

Relations of Numbers and Square Roots of their Powers

The square root of any power of a number is the identical power of the square root of such number.

Take the number, 256, which is the 4th power of 4; its square root is 16, and 16 is the identical power of the square root of 4, because the square root of 4, which is 2, raised to the identical power, the 4th power, is equal to 16.

In the two double columns below, the numbers 4 and 9 are used as the bases for the illustration of this law.

The left hand column of each pair of columns contains the consecutive powers of 4 and of 9, the series being carried in each case up to the 7th power. The right hand columns of each pair contain the square roots of the powers in question. It will be seen that these roots are the identical powers of the square roots of 4 and of 9 respectively.

4	2	9	3
16	4	81	9
64	8	729	27
256	16	6561	81
1024	32	59049	243
4096	64	581441	729
16384	128	4782969	2187

In the left hand column take the cube of 4, which is 64. Its square root is 8, which is placed in the adjoining column in a line with 64. Now just as 64 is the cube of 4, so is its square root, 8, the cube of the square root of the original number, 4. The square root of 4 is 2 and 8 is the cube of 2.

Turning to the left hand column of the powers of 8, we find the 7th power of 9 to be 4782969; its square root is 2187, and 2187 is also the 7th power of the square root of the original number, 9; it is the square root of 3. The same will be found to apply to all the powers in the two columns, and the reader may try other numbers to any extent.

Terminal Digits of Squares of Numbers

No exact square ends in 2, 3, 7 or 8. A number ending in one of the other five digits may be a perfect square; if it ends in one of these four it cannot be. An even square may end in any one of the other digits, 1, 4, 5, 6 or 9 followed by an even number of ciphers. Thus 302500 is a perfect square; its root is 550. This gives a way of telling that a number has no perfect square root; it is a pity that it does not go a little further and tell that a number is a true square, but it does not.

If a square of a number ends in 4 the figure immediately preceding 4, the last figure but one, must be an even number or a cipher. Thus the square of 98 is 9604, the square of 62 is 3844.

If a square ends in an odd figure the figure preceding it must be an even one. The squares of 5, 7 and 9 are examples, 25, 49 and 81 respectively.

If a square ends in any even number except 4, the last figure but one will be an odd one. Thus the square of 16 is 256.

If a square ends in 5 the figure 2 will precede the five; in other words such a square will end in 25. Thus the square of 15 is 225.

A square may end in two ciphers or in two 4's; otherwise it cannot end in a doubled number. The square of 12 is an example of the doubling—it is 144. The square of any number ending in 0 ends in two ciphers.

No square may end in a single cipher or in an odd number of ciphers; it may end in any even number of them.

A square may end in three 4's; it cannot end in any other three figures. Thus the square of 462 is 213444.

The square of a number ending in one or more ciphers ends in twice the number of ciphers in question. The square of 90 is 8100; the square of 700 is 490000.

A Curious Fraction

The fractional quantity, $\frac{41}{12}$, if squared, gives the

fraction, $\frac{1681}{144}$. The latter fraction possesses the curious property that if it is increased or diminished by 5, the sum in the one case and the difference in the other case will both be perfect squares.

Reducing 5 to the fractional form and with a denominator, 144, gives $\frac{720}{144}$, which is in shape to be added or

subtracted from the other fraction. If we add it we have $\frac{2401}{144}$, whose square root is $\frac{49}{12}$, and if we subtract it the

resulting fraction is $\frac{961}{144}$, whose square root is $\frac{31}{12}$.

Circular Numbers

The numbers 5 and 6 are called circular numbers. The property which is appealed to as a basis for the name or title is the fact that their squares, cubes and all other powers end respectively in the numbers, 5 and 6. All powers of 5 end in 5 and all powers of 6 end in 6.

Properties of Squares

Every number squared, as it is, or else when 1 is subtracted from it, is divisible by 3 and by 4. Try such squares as 81, 49, for cases where 1 has to be subtracted; 64, 144, and many others are divisible without the subtraction. Again every squared number is divisible by 5 if it is increased by 1, or if this does not make it divisible then it will be when diminished by 1. Try it on the numbers above and on others.

Some square numbers are odd ones, such as 81, 49 and the like. If from any odd square we subtract 1 the remainder will be divisible by 8. Thus 729 is an odd square number, it is the square of 27; subtract 1 and we have 728, which is 91×8 .

If the sum of the squares of two numbers is a perfect square, then the product of the two numbers is divisible by 6. This would be a very useful property of squares if it only worked the other or both ways—but it does not. Thus the product of 6 and 9 is divisible by 6, but the sum of their squares is 117, which is not a perfect square. Here is where it does not apply.

If the difference of the squares of two numbers is a perfect square the sum and difference of the original numbers are squares or the double of squares. Try it

with 6 and 10, and then with 12 and 13. Thus $10^2 - 6^2 = 64$, which is a perfect square; the sum of 10 and 6 is 16 and the difference is 4, both of which are perfect squares. Such numbers are not particularly numerous.

Write in a line the squares of the natural numbers, 1, 2, 3 and so on. Under them write the series 1, 3, 5, 7... Then the squares will be obtained by adding the series numbers to the squares one by one.

- a. Natural numbers 1, 2, 3, 4, 5,...
- b. Their squares 1, 4, 9, 16, 25,...
- c. The series 1, 3, 5, 7, 9, 11,...
- d. Sum of *b* and *c* 1, 4, 9, 16, 25,....

It will be seen that line *d* and line *b* are identical.

Property of the Square of Two

The number 2 is the only integral number whose square is equal to its double; this amounts to saying that $2 \times 2 = 2 \times 2$, taking one multiplication as the expression of squaring, and the other as the expression of simple multiplication.

Fermat's Last Theorem

What is known as Fermat's last theorem may be expressed as follows: There are no three integral numbers so related that, if all are raised to the same power, and the power is higher than the square, the sum of any two will be equal to the third. We know and have seen that there are any number of quantities the sum of the two squares of two of which are equal to the square of the third, but if we go above the squares of numbers no similar relation can be found to exist.

Properties of Cubes

The following property of the cubes of four successive numbers is analogous to the relation of the squares, $3^2 + 4^2 = 5^2$. The property refers to the numbers 3, 4, 5, and 6, and is expressed as below:

$$3^3 + 4^3 + 5^3 = 6^3.$$

The volume of a cube is equal to the cube of one of its sides. To solve the classic problem of the duplication of the cube by simple arithmetic, which of course was not contemplated in the original problem, we must calculate the length of the edge of the second cube, such that its cube will be twice the cube of the edge of the smaller cube. The relation of the edges must be as $\sqrt[3]{2}$ is to 1. No integral numbers can be found giving this relation.

Orders of Differences of Powers

Suppose a row of numbers is set down on a line, and that on the line below them the successive differences are placed, each difference naturally below the space that comes between the two numbers appertaining to it. This is called the first order of differences, and there will be one less difference than the original numbers. Next if the rows of differences are subtracted from each other the new row of figures is the second order of differences, and it will again be one less in number than its parent row. Now try this with the squares of numbers. The square of 1 is 1, that of 2 is 4 and so on.

Squares	1	4	9	16	25	36	49	64	&c.
First order		3	5	7	9	11	13	15	
Second order			2	2	2	2	2	2	

The second order of differences of successive squares is always 2, no matter how large the squares may be. The difference, 2, is the product of 1, the first and only order of differences between the natural numbers, multiplied by 2, the latter the exponent of the second power or square of any number. For cubes we find that a uniform series of differences is reached in the third order of differences.

Cubes	1	8	27	64	125	216	&c.
First order		7	19	37	61	91	
Second order			12	18	24	30	
Third order				6	6	6	

The difference, 6, is the product of 1 by 2 by 3—the continued product of the three exponents of the first, second and third powers. To get the final difference of the fourth powers of numbers the same process may be carried out, or what is simpler take the continued product of the exponents just as before. This gives for the fourth powers 1 by 2 by 3 by 4, which is 24; for the fifth powers the fifth order of differences will be 1 by 2 by 3 by 4 by 5, which is 120.

Powers in Progressions

The sums of any number of the odd numbers beginning with 1 and going on in regular succession gives a series of square numbers. A number of the additions are given here; they can be carried to any desired extent. The addition must always start with 1 and no odd number can be omitted from the series.

$$\begin{aligned}
 1 + 3 &= 4 \\
 1 + 3 + 5 &= 9 \\
 1 + 3 + 5 + 7 &= 16 \\
 1 + 3 + 5 + 7 + 9 &= 25 \\
 1 + 3 + 5 + 7 + 9 + 11 &= 36 \\
 1 + 3 + 5 + 7 + 9 + 11 + 13 &= 49 \\
 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 &= 64 \\
 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 &= 81
 \end{aligned}$$

All the sums are perfect squares. The same process can be carried out as far as desired. The number of the quantities added tells the square root. Thus the addition of seven quantities gives a number whose square root is 7, and so for all other additions by this method.

Taking the number 3 as a starting point if we add together the first two uneven or odd numbers, namely 3 and 5 we have the cube of the number of digits used. We used the digits 3 and 5; adding them together gives 8; they are two in number and 8 is the cube of 2. Now take the next three and they will give the cube of 3, which is 27, thus: $7 + 9 + 11 = 27$. Now take the next four, and we shall get the cube of four, which is 64, thus: $13 + 15 + 17 + 19 = 64$.

This relationship of the odd numbers to the successive cubes holds good as far as we wish to go.

Here is another curious progression. If the cubes of the natural numbers in their regular order are written out, the addition of the successive cubes will give a series of squares of the series 3, 6, 10, 15, . . .

Natural numbers,	1,	2,	3,	4,	5,....
Their cubes	1,	8,	27,	64,	125,....
Additions, all squares,		9,	36,	100,	225,....
Roots of the additions,		3,	6,	10,	15,....

Thus taking the second line and adding consecutively we find that 1 and 8 give 9 of the line below; then 1 and 8 and 27 of the second line give 36 of the third line and so for the rest or as far as it may be carried, and the roots of the squares of the third line give the fourth line, which is the series referred to in the text.

Relations of Squares of Two Numbers

Take any odd number and divide it into two parts such that they will differ by unity. These two numbers will be the roots of two squares which will differ in amount by the original odd number. Thus take the number 13. It can be divided into two parts, 6 and 7. Squaring these we get 36 and 49, whose difference is the original number, namely 13. The same applies to any odd number whatever; 5 divides into 2 and 3, and the difference of their squares 4 and 9 is 5, the original number.

Many other curious relations can be traced out. Any series of even numbers beginning with 12 and going on with increments of 4, 12, 16, 20, 24 and so on—is to be divided by 2. The number thus obtained is to be divided into two parts differing by 2. These numbers squared will give two numbers whose difference will give the original number. Take 16 for example; divide by 2 giving 8. From 8 we obtain two numbers differing by 2, namely 5 and 3, whose sum is 8. Squaring these we get 9 and 25, whose difference is 16, the original number.

This cannot be done with even numbers indivisible by 4, because such numbers when divided by 2 give odd numbers, and these cannot give the two aliquot parts differing from each other by 2. Try it for yourself.

A well known problem is the following: Find a num-

ber such that if 12 and 25 be successively added to it, the results will be square numbers. It is evident that the difference between the two squares will be $25 - 12$ or 13. This we divide into two numbers differing by one and squaring these we have the numbers sought for. The two numbers into which we divide 13 are 6 and 7, their squares are 36 and 49, and the number asked for in the problem is $36 - 12$ or $49 - 25$, either subtraction giving 24, which is the answer.

The process can be carried to other differences. Take the series of numbers differing by multiples of 5. What squares differ by 35 may be asked? Divide by 5 which gives 7; divide in two parts differing by 5, namely 1 and 6; squaring these gives 1 and 36, two square numbers differing by 35.

The number 35 could have been made to give another answer by treating it as an odd number simply. Thus divide it into two parts differing by 1; 17 and 18; square these which gives 289 and 324, both squares and differing by 35 also.

A general rule is when you have the choice of two numbers to divide with, divide by the smaller. Thus we got an answer by dividing 35 by the smaller number, 5; had we divided by 7, which is the other divisor, and the larger one, we should have failed to get an answer. Also divide even numbers by even ones and odd ones by odd.

The Progression of Cubes

Take the cubes of numbers in their numerical order—the progression of cubes it is called—and add them consecutively, putting down each number as obtained. The results will be perfect squares as below.

Cubes—	1	8	27	64	125	216	&c.
Additions—	1	9	36	100	225	441	
Square roots—	1	3	6	10	15	21	

The set of square roots constitute a true progression with a difference increasing by unity, and the same square roots are the sum of the cube roots of the upper row of numbers. These cube roots are:

1 2 3 4 5 6

and their sums give the row of square roots as above. Thus 1 and 2 are 3, the second of the row of square roots, and 1 and 2 and 3 are 6, the third figure and so on.

A Curious Property of Two Squares

Here is another odd property of squares. Take any two numbers, one above and one below 25 and equally removed therefrom—say 14 and 36. The difference between each of them and 25 is 11. Then the difference of their squares will be 1100 and their squares will end in the same two numbers. We give the calculation for this and two other pairs.

(36) = 1296	(44) = 1936	(29) = 841
(14) = 196	(6) = 36	(21) = 441
1100	1900	400

The first two figures of these remainders are differences between the numbers and 25; 14 and 11 are 25; and 36 minus 11 are 25; and so for the others. The two numbers of the second pair differ by 19 each from 25,

and the numbers of the third pair differ by 4. The reader can try the same calculations with 50 and with 75 as base numbers and see the relations.

The Sum of Two Squares

If the product of 4 by a number is such that the addition of unity thereto produces a prime number, then such prime number and any power of it is the sum of two squares.

Thus take the product of 4×3 which is 12 and add 1 thereto. This gives 13, a prime number, formed by adding unity to a product of 4. Thirteen is the sum of $9 + 4$ both square numbers. Now take a power of 13; $13^2 = 169 = 12^2 + 5^2$, again the sum of two squares. Next take 4×10 and add 1 thereto; this gives 41, a prime number and again the sum of two squares, 5^2 and 4^2 . Next square this: it gives 1681, the sum of two squares, 40^2 and 9^2 ; we may cube it next; $41^3 = 68921$, and this is the sum of two squares, 236^2 and 115^2 .

If two numbers are such that the sum of their squares is also a square, then the product of the two numbers is divisible by 6.

The numbers 3 and 4 have as squares 9 and 16; the sum of these squares is 25, which is also a square; therefore the product of the two original numbers is divisible by 6; this product is $3 \times 4 = 12$, into which 6 goes twice without remainder.

To find two such numbers, the sum of whose squares shall be a square take any two numbers and multiply them together. Twice their product will be one of the numbers, and the difference of their squares will be the other.

Take 4 and 7 as the numbers to start with. Their

product is 28, and twice this is 56; this is one of the numbers to be found. The difference of their squares is $49 - 16 = 33$, which is the other number sought. The square of 56 is 3136; the square of 33 is 1089; the sum of the two figures is 4225, which is a perfect square, whose root is 65.

The Square of the Hypotheneuse

There is one right angle triangle which is very well known, and which has long been used for laying off square work, such as foundations of buildings, fence corners and the like. This triangle is one whose sides are in the proportion of 3 to 4 to 5. Thus for the corner of a field we may take 10 feet as the unit. Then if we measure off 30 feet on a side known to be correct, and then with a 40 foot string scratch an arc of a circle from the corner end of the field as a centre, and next with the outer end of the 40 foot line as a center and a 50 foot string strike an arc of a circle intersecting the other arc, a line from the intersection of the two arcs to the corner of the lot will be a right angle. A square can be extemporized from three laths on the same principle. One lath is cut a little over three feet in length, the others a little over four and five feet respectively. Holes are bored near the ends of the laths, exactly on the axis, and exactly three, four and five feet apart. Joining them by tightly fitting wire nails you will have an accurate square.

The following is an elegant rule for laying out any kind of right angle triangle. Take any two numbers. For one of the sides of the triangle take twice the product of the two; for the other side, the difference of the

squares of the numbers; and for the hypotheneuse the sum of their squares.

Suppose we take 2 and 7 as the numbers. Twice the product of the two is $2 \times 7 \times 2 = 28$. This is one side. The difference of their squares is $49 - 4 = 45$. This is the other side. Then for the hypotheneuse we take the sum of the squares— $4 + 49 = 53$.

By trying different numbers all sorts of right angle triangles can be calculated. It is obvious that the numbers taken must be different ones. Thus try 3 and 7 as the numbers and the resulting triangle will be almost equilateral. 4 and 7 give a triangle whose short side is almost exactly one-half the hypotheneuse.

The formation of the following series of mixed numbers hardly needs explanation. The series is $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$. . . and so on as far as desired. The integral numbers and the numerators of the fractions are successively increased by the addition of 1; the denominators of the fractions are successively increased by the addition of 2. Now if these numbers are converted into improper fractions they will run as follows: $\frac{4}{3}$, $\frac{12}{5}$, $\frac{24}{7}$, $\frac{40}{9}$ These improper fractions give the sides of perfect right angle triangles, that is to say of right angle triangles with integral number sides.

Taking the fraction $\frac{24}{7}$; this gives one side of the triangle as 7, the other side as 40 and the hypotheneuse as the square root of the sum of the squares of 7 and of 40. The sum of these squares is 1681, and the square root of 1681 is 41, which is the hypotheneuse of the triangle in question.

If we try the same with any fraction of the series the sum of the squares of its numerator and denominator will give a perfect square.

Approximations to Isosceles Right Angle Triangles

There are a few cases where the sum of the squares of successive numbers gives a perfect square. Such numbers are 3 and 4, 20 and 21, 119 and 120. There are six more pairs up to the pair 803760 and 803761, it is said. These numbers give the closest approach to isosceles right angle triangles, as no right angle triangle with equal sides exists.

Short Way of Raising to High Powers

If the exponent of a power to which a number is to be raised can be factored, the value of the power can be found by raising the number first to the power indicated by one of the factors, then raising this power to that of another factor until the factors are expended.

Suppose 2 is to be raised to the 9th power. 9 can be factored, it is equal to 3×3 . Therefore we can raise 2 to the cube, which gives 8, and raise 8 to the cube (which is the other factor of 9), giving 512; this is the value of 2^9 or what is the same thing $2^{3 \times 3}$.

Suppose 7 is to be raised to the 12th power. It is much easier to factor the exponent, 12, taking it as $2 \times 2 \times 3$; to raise 7 to the second power, 49; to raise 49 to the second power, 2401; and to cube this, giving the value required, 13,841,287,201, than to multiply 7 the eleven times required to do it directly.

Extraction of Roots of High Powers

The more important application of this principle is in the extraction of roots. If a root whose index or exponent is factorable is to be extracted, all that is necessary

to do, is to successively extract the roots indicated by the factoring.

If the sixth root of a number is to be extracted at one operation it is a long process, involving tentation. But if the square root is first extracted and the cube root of the square root extracted this gives the sixth root without the tedious extraction of a high root.

For the extraction of roots, whose indices are prime numbers this method is inapplicable.

Short Method for Calculating Squares

The following method of calculating the squares of numbers is available up to 100. Beyond that it could still be used by a good computator, but it is especially recommended for numbers under the century mark. It is based on the principle proved in algebra, that the product of the sum and difference of two numbers is equal to the difference of their squares. Thus $29 + 1$ multiplied by $29 - 1$ gives $(29)^2 - 1$. If therefore we multiply 30 by 28, for these are the values of $29 + 1$ and $29 - 1$, and if to their product 1 is added the result will be the square of 29. It is easier to multiply 28 by 30 than to directly square 29.

To square a number proceed as follows:

Add to the number or subtract from it a quantity which will produce a decimal number, such as 20 or the nearest product of units by 10. This gives a single digit to multiply by, which is very easy. Then subtracting the same quantity from the original number, multiply the two new numbers together and to their product add the square of the number added or subtracted.

Square 79.

Add and subtract 1; this gives 78 and 80; $78 \times 80 = 6240$, and adding the square of the increment, which in this case is 1, the square of 79 is the result, 6241.

Square 67.

Here we add and subtract 3, obtaining as our working numbers, 64 and 70. Multiplying 64 by 70 gives 4480, and to this must be added the square of the increment, 3; this square is 9 and $4480 + 9 = 4489$, which is the square of 67.

The clue to the process is by addition or subtraction to get a decimal number for one of the multipliers.

The next example shows the use of subtraction to get the desired decimal number.

Square 54.

Subtracting and adding 4 gives the two multipliers as we may call them, 50 and 58. It will be observed that the decimal number this time is obtained by subtraction. We have therefore $50 \times 58 = 2900$, and adding the square of the increment, 16, we have the square of 54, which is 2916.

Subtraction is used to get the two working numbers when the smaller increment is required to give the decimal number. It is only a matter of convenience whether addition or subtraction is employed.

Short Way of Calculating Squares of Large Numbers

The square of the sum of two numbers is proved in algebra to be equal to the sum of the square of the first added to the square of the second and to twice the product of the first by the second. The law is best illustrated by a couple of examples.

$$(93)^2 = (90 + 3)^2 = (90)^2 + 3^2 + 2(90 \times 3) \\ = 8100 + 9 + 540 = 8649$$

$$(24)^2 = (20 + 4)^2 = (20)^2 + 4^2 + 2(20 \times 4) \\ = 400 + 16 + 160 = 576$$

By a little practice this method becomes very easy of execution, and it can often be used to advantage as a method of calculating squares or as a way of proving results.

When applied to mixed numbers, such as $5\frac{1}{2}$, $6\frac{1}{4}$ and the like, a general rule may be thus stated: Add to the integral number double the fraction, and multiply this by the integral number; add to the sum the square of the fraction.

To calculate the square of $9\frac{1}{3}$ by the above rule: Add to the integral number, which is 9, twice the fraction, giving $9\frac{2}{3}$; multiply this by 9; this gives $9 \times 9\frac{2}{3} = 87$; adding the square of the fraction, $\frac{1}{9}$, we have $(9\frac{1}{3})^2 = 87\frac{1}{9}$.

This squaring is particularly easy because 9 is divisible by 3. If $8\frac{1}{3}$ is raised to the square by this process it will not be quite as simple, although there is no difficulty about it; thus: $8 \times 8\frac{2}{3} = 69\frac{1}{3}$, and adding the square of the fraction, which is $\frac{1}{9}$, gives $69\frac{1}{3} + \frac{1}{9} = 69\frac{4}{9}$, which is the square in question.

Aliquot parts apply often here. 625 may be taken as $6\frac{1}{4}$. The method is applied with due regard to the decimal point; first we get $6 \times 6\frac{1}{2} = 39$; to this is added the square of $\frac{1}{4}$ which is $\frac{1}{16} = .0625$, and the final result is 39.0625, for the decimal point disappears as the original number contained none.

Various Ways of Squaring Numbers

If a number is divisible by 2, by 3 or by 5, its squaring may be simplified in many cases by dividing it by one of these numbers, squaring the quotient and multiplying by the square of 2 or of 3 according to which was the divisor we employed.

To square 33; dividing by 3 and squaring the quotient gives 121; this must be multiplied by 3^2 , which is 9; $121 \times 9 = 1,089$.

To square 36, proceed exactly as in the last case; $36 \div 3 = 12$; squaring 12 gives 144, and this multiplied by 9 gives 1,296.

Now take a number ending in 5—say 45. Dividing by 5 gives 9, and squaring 9 gives 81, and $81 \times 25 = 2,025$. Here of course we multiply by 100 and divide by 4; it can be done mentally and put down directly on the paper.

If the square of a number is known the square of the number which is one greater than itself, is found by adding to this square its own square root and the number which is one greater than its own square root.

Thus to determine the square of 13; the square of 12 is 144; to it add its own square root, 12, and the number which is one greater, 13; this gives $144 + 12 + 13 = 169$, which is the square of the number one greater than 12, or the square of 13.

The product of two numbers increased by the square of half their difference is equal to the square of the intermediate number.

Thus the product of 24 and 25 is 600; their difference is 1, and the square of $\frac{1}{2}$, or the square of half their difference, is $\frac{1}{4}$, the square of the intermediate number is $600\frac{1}{4}$; this number is $24\frac{1}{2}$.

Again the product of 20 by 80 is 1600; half the difference is 30; if its square is added to 1600, it gives $1600 + 900 = 2500$, which is the square of the intermediate number, or the square of 50.

To square a number ending in $\frac{1}{2}$ or what is the same thing for this method, one ending in 5, proceed as follows:

In the first case multiply the integral portion of the number by the next higher integral; add to the product $\frac{1}{4}$, and the result is the square of the original mixed number.

Thus the square of $8\frac{1}{2}$ is $(8 \times 9) + \frac{1}{4} = 72\frac{1}{4}$.

If the number ends in 5, the 5 is taken as representing the $\frac{1}{2}$ of the first case; thus the square of 65 is $(60 \times 70) + 25 = 4225$.

This rule is often very convenient.

To square any number between 25 and 75, subtract 25 from the number, multiply the remainder by 100 and add the square of the difference between the number and 50.

Thus to square 46 proceed as follows:

$$\begin{array}{rcl} 46 - 25 = 21, \text{ and } 21 \times 100 = 2100 \\ 50 - 46 = 4, \text{ and } 4^2 = 16 \\ \hline 46^2 = 2116 \end{array}$$

To square any number made up entirely of 9's, proceed thus:

Put down 1 for the right-hand figure; on its left put ciphers one less than the 9's in the original number; next put an 8, and finally 9's, one less in number than the 9's in the original number. By this rule we can write out the square of any such number at once; thus the square of 9999 is 99980001; the square of 99999 is 9999800001.

McGiffert's Methods of Squaring Numbers

The following method of squaring numbers between 40 and 60 is due to Prof. James McGiffert of the Rennselaer Polytechnic Institute.

The difference between the number and 50 is called the complement. If the number exceeds 50 then the complement is added to 25 and the square of the complement is annexed, not added, to the result. Suppose 57 is to be squared. The difference between 57 and 50 is 7, this is the complement. Carrying out the rule we have: $25 + 7 = 32$; annexing the square of the complement or 49 to 32 we have as the square of 57 the number 3249.

If the number is less than 50 the complement is subtracted and the square is annexed as above. Thus the complement of 44 is 6. $25 - 6 = 19$; the square of the complement is 36; annexing this to 19 we have 1936, the square of 44.

Except for the one variation (addition or subtraction) the processes are identical and are based on the rule for the square of the sum and difference of numbers—for the squaring of binomials, to use an algebraic term. The rule will operate on any numbers but it is most advantageous within the range given. The reason it comes so nicely is because one of the numbers is always 50, whose square is 2500 and the second number, namely the complement, is a monomial which is a number less than 10, whose square is in the multiplication table, i. e. whose square is known to everyone. Let us hope so.

If a number is doubled the square of the new number will be four times that of the original. By multiplying any number between 21 and 29 by 2, we can apply the

above rule, and then dividing by 4 will give the square of the original number. Thus take 27. Twice 27 is 54. Then $25 + 4$ with 16 annexed is 2916, and this divided by 4 gives 729, the square of 27.

If we divide any even number from 82 to 98 by 2 and apply the rule, it will give us one quarter of the square of the original number.

This process can be extended to apply to all numbers under 100, if the calculator will only learn the squares of the numbers from 13 to 24, if the doubling process be used to supplement the method for numbers under 25, and the halving process for numbers above 60. Professor McGiffert says these squares are easily learned. The reader is advised to try it.

For the square of a number between 100 and 110, add the complement to the number and annex the square of the complement. Of course if you have learned the squares as just recommended this rule will apply to any number above 100.

Take 109; the complement is 9. Add this to the number— $109 + 9 = 118$; to this annex the square of the complement— $9^2 = 81$ —and we have 11881, as the square of 109, which is correct. If the square of the complement has only one digit a cipher must go before it, between it and the sum of the number and its complement, thus: $103^2 = 103 + 3$ with 3^2 preceded by 0 annexed, giving 10609 as the square of 103.

Between 90 and 100 the same rule applies except that you are to subtract the complement from the number. Thus to square 95 subtract the complement which is 5 and annex the square of the complement, 25, giving 9025, the square of 95.

For numbers above 250, add the complement, annex a cipher and divide by four, and annex the square of the complement. $259 = \frac{(259 + 9) \times 10}{4}$ annex 81 giving

67081. Multiplying by 10 is the same as annexing a cipher.

For numbers below 250 proceed as above but subtract the complement.

If the square of the complement contains three figures be sure to set it only two places to the right. Take 61 whose complement is 11. Following the first rule we add the complement to 25 giving 36; the square of the complement is 121, which is annexed with the 1 under

the 6 of 36, thus: $\begin{array}{r} 36 \\ 121, 3721 \end{array}$ is the square of 61.

Prof. McGiffert's methods it will be seen apply to all numbers, and are very interesting and practical.

Nexen's Method of Extracting Higher Roots

The following method of extracting the cube and other high roots is easier than the regular method of the arithmetics, and is also of interest, because, by carrying out the same principles, any root may be extracted; examples are given of the extraction of the 11th root in Prof. Nexen's book. The cube root extraction follows here.

Point off the number in threes as usual and by inspection and trial if necessary get the nearest root of the first period pointed off. This root may be either too large or too small, but it must be the nearest.

Raise this approximate root to the power represented by half of the exponent or if the exponent is an odd number, raise it to the power of what may be called the "larger half" of the exponent. For a cube root, where the exponent is 3, the "larger half" is 2; for the fifth root it would be 3.

Divide the original number by this quantity; the quotient will be an approximation to the cube root sought. Add to it twice the original approximate root and divide by 3; this gives an average which is a still closer approximation to the correct root.

With this new approximate root repeat the operation and so *ad libitum*, until a close enough result is obtained. Two operations are close enough for all ordinary purposes.

The rule seems long, but the operation is very short as will be seen in the examples given here.

To extract the cube root of 250, proceed as follows: A mental trial shows that 6 is the nearest integral root of 250; it is raised to the second power in accordance with the third paragraph of the rule, and the number is divided by it. We have therefore

$$\begin{array}{r}
 6 = 36; \quad 36 \overline{) 250} \quad (6.94 \\
 \underline{216} \\
 340 \\
 \underline{324} \\
 160
 \end{array}$$

The quotient, 6.94, is an approximation to the root sought for; it is added to twice the first approximate root,

which was 6, and the average of the three is squared and used as a new divisor, because it is a closer approximation to the true root than either of the others.

$$\begin{array}{r} 6 \times 2 = 12 \\ 6.94 \\ \hline \end{array}$$

18.94 average is 6.31 and $6.31^2 = 39.82$ the next divisor.

$$\begin{array}{r} 39.82 \) \ 250.000 \ (\ 6.2782 \\ 238 \ 92 \\ \hline \end{array}$$

&c.

$$\begin{array}{r} 6.31 \times 2 = 12.62 \\ 6.278 \\ \hline \end{array}$$

18.898 average is 6.299

and $6.299^2 = 39.677$ is the next divisor and the quotient of 250 divided by 39.677 is 6.300. Using this to get an average exactly as in the preceding cases, we have:

$$\begin{array}{r} 6.299 \times 2 = 12.598 \\ 6.300 \\ \hline \end{array}$$

18.898 average is 6.2993

If required we can go on as far as we desire. But comparing the three approximations so far obtained they are so close that it is not worth while to go further for any ordinary calculation. The three approximations with their cubes are the following:

6.31	whose cube is	251.23
6.3063	" "	250.79
6.2993	" "	249.96

We now will extract the cube root of 377. The cube of 6 is 216, the cube of 7 is 343, the cube of 8 is 512;

the nearest root is therefore 7. We now proceed as before, but the operations will be merely indicated.

$$\begin{array}{r}
 7^2 = 49; \quad 49 \overline{) 377} \quad (7.69 \quad 7 \times 2 = 14 \\
 \underline{343} \qquad \qquad \qquad \underline{7.69} \\
 \text{\&c.} \qquad \qquad \qquad 21.69 \quad \text{average is } 7.23 \\
 7.23^2 = 52.2729; \quad 52.2729 \overline{) 377.00000} \quad (7.212 \\
 \qquad \qquad \qquad \underline{3659103} \\
 \qquad \qquad \qquad \text{\&c.} \\
 7.23 \times 2 = 14.26 \\
 \underline{7.212} \\
 21.672 \quad \text{average is } 7.224 \\
 7.224^2 = 52.186; \quad 52.186 \overline{) 377.0000} \quad (7.224 \\
 \qquad \qquad \qquad \underline{365302} \\
 \qquad \qquad \qquad \text{\&c}
 \end{array}$$

Here we have reached the root because the number, 377, divided by the square of 7.224 gives a quotient which is the same, namely, 7.224.

The approximation to the exact cube root attained at any step of the calculation can be judged by comparing the number whose square is used as the divisor, with the quotient given by dividing the original number by the square in question.

The above result is given by logarithms also.

To extract the fifth root of a number, the approximate fifth root is the basis for the first division. This is raised to the cube and the quantity whose root is to be extracted is divided by it. The quotient is the approximate square

of the fifth root sought for. So we take the average obtained by dividing the sum of three times the cube root of the first divisor and of twice the square root of the quotient by 5 and repeat the operation with this as a divisor. By repeating the process three or four times a very close approximation to the exact fifth root is obtained. The process of determining the fifth root of 399 is shown here in outline, operations being indicated and results given.

As the approximate fifth root we start with 3, and divide by its cube which is 27.

$$\begin{array}{rcl}
 3^3 = 27; & 399 \div 27 = 14.77; & 3 \times 3 = 9 \\
 \sqrt{14.77} = 3.85 & & 3.85 \times 2 = 7.7 \\
 & & \hline
 & & 5 \mid 16.7 \\
 & & \hline
 & & 3.34 \\
 3.34^3 = 37.26; & 399 \div 37.26 = 10.7084; & 3.34 \times 3 = 10.02 \\
 \sqrt{10.7084} = 3.29 & & 3.29 \times 2 = 6.58 \\
 & & \hline
 & & 5 \mid 16.60 \\
 & & \hline
 & & 3.32 \\
 3.32^3 = 36.59; & 399 \div 36.59 = 10.9046; & 3.32 \times 3 = 9.96 \\
 \sqrt{10.9046} = 3.302 & & 3.302 \times 2 = 6.604 \\
 & & \hline
 & & 5 \mid 16.564 \\
 & & \hline
 & & 3.3128
 \end{array}$$

The fifth root given by logarithms is 3.3125, practically identical.

Extractions of the fifth root by regular arithmetical process is a very long and tedious operation. The above method can be carried out to any desired degree of accuracy.

This method can be used for the square root, although the regular method is, to the writer's mind, more satisfactory. Let the square root of 2981 be required.

As the first divisor we take 5, the approximate square root of 29, the first two-figure period of the number in question.

$$2981 \div 5 = 5962;$$

In arranging the addition for the average we must take into account the fact that the divisor, 5, was referred to the first period, so in addition we put the two figure 5's under each other exactly as if the rest of the quotient was a decimal fraction.

$$\begin{array}{r} 5 \\ 5 \ 962 \\ \hline 2 \) \ 10 \ 962 \\ \hline 5 \ 481 \end{array}$$

We now do the second division.

$$2981 \div 548 = 544.$$

The average of the divisor and quotient ($548 + 544$) $\div 2 = 546$, which is the square root as far as the figures are concerned, but as there are four figures in the original number the decimal point must go after the second figure, so that the root is 54.6.

The regular calculation carried out to four places of decimals gives 54.5985, practically the same. By cutting off no decimals and carrying the operation through some more divisions the root could have been brought to any degree of accuracy desired.

The above methods are interesting and instructive and practical for the cube and high roots, but where a higher root with a prime number for the exponent is to be extracted it is best to do it by logarithms.

CHAPTER X

EXPONENTS

Exponents of Powers

If a number greater than unity is multiplied once by itself, it is said to be raised to the second power; if multiplied by itself twice, it is said to be raised to third power. The same nomenclature is carried out for all powers. This system is not inconsistent as it may seem on first sight. The numerical value of the power, expressed by the exponent, does not directly indicate the number of multiplications involved in raising the number to the given power; it indicates the number of times the number raised enters into the multiplication. To raise a number to the second power, as it is multiplied once by itself, it is obvious that it will enter twice into the multiplication; if raised to the third power it will enter three times and so on.

Any power of a number, such power being expressed by an integral positive number, contains the original number as factors as many times as the exponent of the power indicates.

The exponent of a power may be a positive or a negative integral or fractional number; if it is an improper fraction it may be written as a mixed number if desirable.

Suppose that the quantity, 2, is to be raised to the fifth power, whose exponent is 5. It must be multiplied four times by itself, thus: $2 \times 2 \times 2 \times 2 \times 2 = 32$.

Here there are four multiplications but five equal factors, each one the original number; the five factors determine the power as the fifth power, 2^5 , in numerical value, 32.

Fractional Exponents

A fractional exponent with 1 for numerator indicates the root of the number corresponding to the denominator of the fraction.

Thus the number, 4, to the half power, in figures, $4^{\frac{1}{2}}$, means the square root of 4, which is 2; 81 to the one-fourth power, $81^{\frac{1}{4}}$, means the fourth root of 81, which is 3.

A fractional exponent may have any number for its numerator. The number affected by such an exponent, if the indicated operation is carried out, is to have the root indicated by the denominator of the fraction raised to the power expressed by the numerator. This may be put the other way, and the exponent may be taken as indicating the same root of the same power of the original number. Both statements mean the same in effect.

Thus 8 to the two-thirds power, $8^{\frac{2}{3}}$, means the cube root of the square of 8; the square of 8 is 64, and the cube root of 64 is 4, so $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

This could have been taken the other way. The cube root of 8 could have been raised to the square; the cube root of 8 is 2, and 2 raised to the square is 4.

The first process is generally the better one.

If the fractional exponent or index of the power is an improper fraction, the same method of treatment holds in all respects. But it is possible to introduce a somewhat different method by converting the improper

fraction into a mixed number. Before doing this a rule must be stated.

An exponent may consist of the sum of several numbers. To carry out the indicated operation, we may raise the number to the different indicated powers, multiply one by the other, and the continued product of all will give the value. Thus 4^{2+2+3} is the product of $4^2 \times 4^2 \times 4^3$ or $16 \times 16 \times 64 = 16,384$. We may add the exponents and raise the original number, 4, directly to the power indicated by their sum, which is 7; or $4^7 = 16,384$.

Returning to the consideration of a mixed number exponent, a mixed number is the sum of an integral number and a fraction. Therefore if we are to reduce a number with a mixed number exponent, we may separate the exponent into two parts, an integral and a fractional part, calculate each separately, and multiply them together.

What is the value of $9^{3\frac{1}{2}}$?

$$9^{3\frac{1}{2}} = 9^{1\frac{1}{2}} = 9^{1+\frac{1}{2}}, \text{ and } 9 \times 9^{\frac{1}{2}} = 9 \times 3 = 27.$$

Operating directly with the fractional exponent, we raise 9 to the cube giving 729, and extract the square root of 729, which gives 27 as before. The result is the same in both cases; there are two ways of doing the same thing.

We now come to another rule concerning exponents.

An exponent may be indicated by the subtraction of one number from another. In such case the original number may be raised to the power represented by the numerical difference between the exponents, with the sign, + or —, of the larger exponent prefixed to the difference. Thus 4^{3-1} may be expressed as 4^2 ; 4^{1-3} may

be expressed as 4^{-2} . But we may, if we wish, raise the number to each partial exponent, and divide the one with the positive exponent (the minuend) by the other.

Taking 4^{3-1} as before we may divide 4^3 by 4^1 or 64 by 4, giving 16; or we may say $4^2 = 16$, $4^{3-1} = 4^2$ and the answer, both results being identical.

Next take 4^{1-3} ; following the rule we divide 4 by 4^3 ,

which gives $\frac{4}{4^3} = \frac{4}{64} = \frac{1}{16}$, as before. And we may say

$$4^{1-3} = 4^{-2} = \frac{1}{16} \text{ as before.}$$

In the present book algebra has been avoided, as our object is to treat the subjects on a strictly arithmetical basis. A slight departure is inevitable here, as a negative exponent falls within the scope rather of algebra than of arithmetic.

Zero as an Exponent

Assume any number to have two equal exponents, one subtracted from the other. Such would be 2^{2-2} , 3^{3-3} , 9^{2-2} , and the like. We have seen that when a number has two exponents separated by the minus sign, as in these cases, the value of the expression is obtained by dividing the number raised to the first exponent by the same number raised to the second one. But here both exponents are the same; therefore the number raised to the power represented by one exponent will be the same as for the second exponent. Thus $2^{2-2} = 4 \div 4 = 1$. The same applies to 3^{3-3} , which gives $27 \div 27 = 1$, and $9^{2-2} = 81 \div 81 = 1$. Now we must note the value of the exponents of the numbers; the value of $2 - 2$ is 0;

that of $3 - 3$ is also 0. So we find that these numbers are of the zero or 0 power, and that any number of the 0 power is equal to unity.

Thus 5^0 , 100^0 , $\frac{1}{2}^0$ or any quantity affected with the exponent 0, is equal to 1.

This is in line with the converse, that unity is always equal to unity, whether an exponent is given it or not; $1^2 = 1$, $1^4 = 1$, and so for all exponents.

Prime Exponents

The following property of prime exponents is attributed to a Chinese source.

If 2 is raised to any power, whose exponent is a prime number, and if 2 is subtracted from it when so raised, the result will be divisible by the exponent used without a remainder; the quotient will be an integral number.

Take 3, a prime number, as the exponent of 2; this gives 8; from 8 subtract 2 and 6 is left, which is divisible by 3, the exponent used. The same may be tried with 5, also a prime number as the exponent. $2^5 = 32$, and $32 - 2 = 30$, which is divisible by the exponent used, which was 5.

Negative Exponents

A negative exponent indicates division of unity by the number with the negative exponent, after such number has been raised to the power indicated by the exponent. Thus 2^{-3} indicates division of unity by 2^3 or by 8, and

putting it in fractional form we have $2^{-3} = \frac{1}{2^3}$ or what is the same thing, $\frac{1}{8}$.

The reason for this is the following:

We have $\frac{2^5}{2^2} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} = \frac{2 \times 2 \times 2}{1} = 2^3$,
 or $2^5 \div 2^2 = 2^3$. Now by the same logic if $2^5 \div 2^2 = 2^3$,
 then $2^2 \div 2^5 = 2^{-3}$; but

$$2^2 \div 2^5 = \frac{2^2}{2^5}$$

and

$$\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2} = \frac{1}{2^3}.$$

$$\text{Therefore } 2^{-3} = \frac{1}{2^3}.$$

Another rule refers to the division of an exponent. If an exponent of a number is divided by a number, the extraction of the root, expressed by the divisor, of the original quantity raised to the indicated power gives the value.

Suppose we have $4^{6 \div 2}$. We have: $4^6 = 4096$. Now as 6 is divided by 2, the square root of 4^6 , which is 4096, must be extracted. $\sqrt{4096} = 64$.

It will be seen that the above can be done by fractional exponents, because $4^{6 \div 2} = 4^{\frac{6}{2}} = 4^3 = 64$.

The number 10, affected by exponents, positive, negative and fractional is used a great deal in electrical calculations, and is applicable in many other fields.

Powers of Ten

The units of electrical measurement are based on the centimeter, gram and second. They are far too large or

too small for practical use; accordingly a series of units called practical units have been adopted and are the only ones used in regular work. It is in the expression of the relation of these practical units to the C. G. S. units as the others are called, from the initials of centimeter, gram and second, that the number 10 affected by various coefficients is employed.

The practical unit of resistance, the ohm, is equal to 1,000,000,000 C. G. S. units of resistance. If the number 10 is multiplied by itself 8 times, it will enter as a factor into the operation, nine times; hence it will be raised to the ninth power, so that 10 with an exponent, 9, will represent the one thousand millions expressed above in figures; it is 10^9 .

The exponent of any power of ten indicates the number of ciphers which come after the 1 in the numerical representation of that power. The above number contains nine ciphers to the right of the 1; therefore the number is the ninth power of ten and is expressed by the numeral ten with an exponent nine— 10^9 .

The volt is the practical unit of potential; it is equal to one hundred millions times the C. G. S. unit of potential; this factor is expressed by 10 with an exponent 8; it is 10^8 .

The ampere is one tenth of a C. G. S. unit of current; this factor or multiplier is written with a negative exponent of 1, thus 10^{-1} , meaning $\frac{1}{10}$.

The unit of capacity is one one thousand millionth of the C. G. S. unit; this gives the multiplier a negative exponent of 9; it is 10^{-9} . But this practical unit is still inconveniently large; it is made one millionth this value by raising the negative exponent of ten to the—15th

power. But instead of having to write down a 1 followed by fifteen ciphers, all that has to be done is to change the exponent, so that the multiplier will be 10^{-15} .

When two or more numbers have identical exponents they can be multiplied by simply adding the exponents together and putting down the original number with the new exponent. This is rather an indication of multiplication than a real multiplication. Applying this rule to powers of ten we have such results as the following:

$$10^{10} \times 10^3 = 10^{13}; \quad 10^2 \times 10^4 = 10^6; \quad 10^7 \times 10^8 = 10^{15}.$$

If the exponents have different signs the difference between them must be taken for the new exponent, and it is to have the sign of the larger numerical exponent affixed to it. Some examples follow:

$$10^5 \times 10^{-3} = 10^2; \quad 10^{15} \times 10^{-10} = 10^5; \quad 10^1 \times 10^{-2} = 10^{-1}.$$

To divide numbers with different exponents, but numerically identical, subtract the exponents algebraically, as in the following examples.

$$10^2 \div 10^3 = 10^{-1}; \quad 10^5 \div 10^3 = 10^2; \quad 10^2 \div 10^{-4} = 10^6.$$

The subtraction is to be done algebraically; accordingly in the first and last example, following the laws of algebra, the sign of the subtrahend was changed and the two exponents were added; this constitutes algebraic subtraction. In the other cases the operation was done arithmetically.

CHAPTER XI

SQUARING THE CIRCLE

Squaring the Circle

A long historical treatise could be written on the subject of the relation between the diameter and circumference of the circle. The ancient term "Squaring the Circle" expresses one of the world's most famous problems, the finding of the length of the side of a square equal in area to a circle of a given diameter. To determine it, the one thing necessary to be known is the ratio of the diameter to the circumference of a circle. If that is known the rest is simple everyday arithmetic. So the term squaring the circle is rather antiquated.

One of the first things many of us were taught in the line of practical mathematics is that once around a circle is three times through or across it. This would be most convenient and is easy to remember, but has one trouble, it is not so. It is the merest approximation to the truth; it is so far from the truth that it should never be used.

Approximations of Former Times

In ancient times the Babylonians and the Jews are supposed to have used this incorrect figure, 3, as the multiple to produce the circumference from the diameter. The ancient Egyptians are credited with doing better than this, with the fractional expression $256/81$ which reduces to 3.1605, a very fair attempt at the value.

Archimedes held that it was less than $3\frac{1}{7}$ and more than $3\frac{10}{71}$; this puts it between 3.1428 and 3.1408, which is correct as far as it goes.

Ptolemy called it, in degree measurement, $3^{\circ} 8' 30''$, reducing to $3 + \frac{8}{60} + \frac{30}{3600}$, or in decimal notation, 3.1416, which is very nearly right.

The Roman surveyors are supposed to have used 3 sometimes and at other times 4, and, when inclined to come a little nearer to the truth, to have used $3\frac{1}{8}$ as the factor.

To Asia is attributed the fraction, $\frac{49}{16}$ or 3.1416, and the quantity expressed as $\sqrt{10}$, which reduces to 3.1622, a very poor attempt at accuracy.

Metius' Value of π

In the year 1585 a celebrated mathematician, Metius, placed the missing number between $\frac{377}{170}$ and $\frac{333}{106}$ and, as it is surmised, perhaps by a lucky guess, found his famous fraction, $\frac{355}{113}$, giving the decimal 3.14159292, which is correct to six places. This is quite close enough for anything short of astronomical dimensions.

Shaw's Value

By the calculus a correct method of calculating it to any degree of accuracy, for it can never be exactly expressed, has been determined, and it has been carried out to most wonderful lengths by computators; one, William Shaw, carried it out to 707 places of decimals.

Geometrical Approximation

A very curious geometrical approximation is obtained by inscribing a square in a circle. If to three times the

diameter of the circle one fifth of the side of the square be added, it will give the circumference of the circle with an error of only $\frac{17}{100000}$. The diameter of the circle is equal to the diagonal of the square inscribed within it; the side of the square, taking its diagonal as 1, is the square root of one half, which is .7071 and this divided by 5 gives .1414, which is .002 less than the true decimal carried to the number of places indicated.

The letter by which the number in question is indicated is the Greek letter, π , pronounced *pi*; it is the Greek *p*, and is supposed to stand for perimeter.

Aid to Memorizing Pi

The following sentence is given as a *memoria technica* for the factor 3.1416 or π carried to the fourth decimal place. The number of letters in each word gives the digits of the factor.

3 1 4 1 6
Yes, I have a number.

Another rhyme by which to remember the value of π , carried out to the twelfth decimal place is the following; as before, the number of letters in each word gives the corresponding digit.

3 1 4 1 5 9
See I have a rhyme assisting
2 6 6 3 5 9 9
My feeble brains its tasks sometimes resisting.

The mathematical value is better than the grammatical.

Curious Determination of π by Probabilities

Buffon and Laplace determined that if a stick of a length less than the distance apart of a series of parallel lines was dropped upon the surface marked with such lines, the probability that it would fall across one of the lines is expressed by the formula $2L/\pi A$, in which L is the length of the stick, which must be less than the distance apart of the lines, or $L < A$, and A is the distance apart of the lines. Sometimes a board floor with boards of equal width is cited as the ruled area.

The experiment of throwing a stick on such a surface has been tried, with results curiously near the true value of π . In one case the stick was thrown 3,204 times and gave the value, 3.1553 for π ; another trial of 600 throws gave 3.137 and in the third case, 1,120 throws gave 3.1419 for π .

The reader is referred to A. de Morgan's *Budget of Paradoxes*, London 1872; and in the *Messenger of Mathematics*, Cambridge, Eng. 1873, Vol. ii, pp. 113, 114.

Squaring the Circle Literally

Having the multiplier, 3.1415927... with which to obtain the length of the circumference from that of the diameter, we can square the circle; this is to find the side of the square equal in area to the circle of any given diameter. The area of a circle is equal to the product of the square of the radius by π . For a circle of the diameter, 1, this area is .7853982 The side of a square of this area is equal to the square root of the same number; this root is .8862. This number is the

length of the side of a square of the area of a circle of diameter, 1.

It will be sufficient to give the following as the value of π ; it is far more accurate than will ever be required by our readers. It is: 3.1415926535

The error here is about one-thirty billionth, not far from an error of sixteen feet in the distance of the sun from the earth.

CHAPTER XII

MISCELLANEOUS

Prime Numbers

A prime number is one which is indivisible except by 1 and by itself. 2, 3, 5, 7, 11 and 13 are prime numbers. Excepting the number 2 they cannot be even; no prime number can terminate in 5, except 5 itself; it follows that after the first period of ten is passed a prime number can only end with a 1, 3, 7 or 9.

Properties of Prime Numbers

A curious property of prime numbers is that, except 2 and 3, if any one of them is increased or diminished by 1, one of the results will be divisible by 6. 101 is a prime number. Diminished by 1 it gives 100, which is indivisible by 6. But if the 1 be added we have 102 which is divisible by 6, as it is the product of 6 and 17. Take next the prime number 73; increase it by 1 and we have 74, which is indivisible by 6, but subtracting 1 gives 72, which is divisible. Take 23, a prime number, add 1, and we have 24, which is divisible by 6.

The product of two prime numbers can never be a square number.

Beginning with 2 and ending with 9973 Hutton gives a list of the prime numbers between these limits; their total is 1139.

The largest prime number known up to the present time is expressed as $2^{61} - 1$; it contains 19 digits.

How to Find Prime Numbers

To find the prime numbers proceed as follows:

Write out the odd integral numbers, 3, 5, 7, 9 and so on as far as desired. Cross out every third number after 3, for none of these are prime. Then cross out in addition to these every fifth number after 5, and then every seventh number after seven and so on. It becomes excessively laborious after high numbers are reached.

If we write out the odd integrals, 3, 5, 7, 9, 11, 13, &c we will find that the numbers eliminated by counting every third one from 3, are 9, 15, 21, 27, 33 Those eliminated by counting every fifth one from 5 are 15 (already eliminated by 3), 25, 35, . . . This leaves as prime numbers 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. If the seven elimination be applied it will only touch up to this point the numbers 21 and 35, both of which have been eliminated, one by the three elimination (21) and one by the five elimination (35).

Perfect Numbers

The aliquot parts of a number are factors or integral numbers which multiplied by other integers give the original number. The number 1 is included. Thus the number six has as its integral parts or numbers, 1, 2 and 3. These three numbers multiplied respectively by 6, by 3 and by 2 give 6 as the product in each case.

A number the sum of whose aliquot parts gives the number in question is called a perfect number. If the aliquot parts of 6 are added together, we have $1 + 2 + 3 = 6$; therefore 6 is a perfect number.

The next perfect number in the numerical order is 28.

Its aliquot parts are 1, 2, 4, 7 and 14. Each of these is an aliquot part, because it will give 28 when multiplied by the proper number. Another way to put it is to say that an aliquot part of a number is a factor of the number; it is any number, which can divide it without a remainder. Now if these five numbers are added together we will obtain the original number, 28; therefore 28 is a perfect number.

To find all the perfect numbers take the geometrical progression, 2, 4, 8, 16, 32, . . . carrying it as far as desired. From this series select those numbers, which diminished by unity give prime numbers. Such are 4, 8, 32, 128 and 8192, if the series is carried out far enough to give it. Diminished by unity these numbers become 3, 7, 31, 127, 8191. Each is now to be multiplied by the geometrical progression which preceded the one from which it is derived. Thus 3 is derived from 4 of the original series, and 4 is preceded by 2 in the series, so 3 is multiplied by 2, giving 6, the first of the perfect numbers. Following out the rule 7 is to be multiplied by 4, 31 is to be multiplied by 16 and so on as far as desired. The result is very meager for there are very few perfect numbers; up to 10^{11} Hutton gives only eight of them.

All perfect numbers or nearly all end in 6 or in 28.

Amicable Numbers

Amicable numbers are numbers so related to each other that the sum of the aliquot parts of one of them gives the other number.

If we take the number 220, its aliquot parts are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; the sum of these is 284. The aliquot parts of 284 are 1, 2, 4, 71 and 142,

and the sum of these is 220; therefore the two numbers, 220 and 284 are amicable numbers.

Other amicable numbers are 17296 and 18416; after these come 9363584 and 9437056. It will be seen that they are excessively rare.

To calculate them start with the same geometrical progression as in the determination of perfect numbers. Call it series A. Multiply each term by 3 and write it under the one from which it was derived. Call this series B. Multiply each number of series B by the number preceding it in the same series and diminish it by unity. Write these down each under the one from which it was derived. Call this series C. The following are the series obtained.

Series A.	2	4	8	16	32	64.....
Series B.	6	12	24	48	96	192.....
Series C.		71	287	1151	4607	18431.....
Series D.	5	11	23	47	95	191.....

Subtract 1 from each figure of series B, and write it under the number from which it was derived. Call this series D.

Take any prime number of series C, subject to this condition: the number under it in series D, and the number preceding this one in series D, must be prime numbers. Multiply the two numbers of series D together, and multiply their product by the number in series A directly over the larger one. Thus in series C, 71 is a prime number; so are 11 and 5 of series D. Following the rule we have $5 \times 11 \times 4 = 220$, one of the numbers. The other is obtained by multiplying 71 by the same number in series A; $71 \times 4 = 284$; this is the corresponding amicable number.

Law of the Square and the Cube

Everything else being equal the larger a cannon ball the farther will it go with the same initial velocity. The reason is based on what may be called the rule of the square and the cube. The resistance offered by the air to the passage of the projectile is due to the area of its cross-section and to what is called skin friction. The area of the section of two similar solids varies with the square of any corresponding linear dimension. This is the law in the case of all surfaces, plane or flat or curved as long as they are similar. A circle for instance varies in area with the square of its diameter. The cross-section of a cannonball is a circle so the same rule applies. The resistance to its flight due to the cross-section therefore varies with the square of its diameter. As regards skin friction the same rule applies; the area of the surface of the projectile varies also with the square of the diameter. Now the force which keeps the ball in motion is its inertia and this depends on its weight and is proportional thereto. But the weight of a body varies with its cubic contents and in the case of two solids of similar shape the cubic contents vary with the cube of any corresponding linear dimension. We may use the diameter; we have therefore the resistance varying with the square of the diameter and the resistance with the cube of the same. The cubes of any two numbers vary more than do the squares; therefore an increase of size will increase the weight more in proportion than it will increase the surface or the cross-section; the larger shell will go further than the smaller one as it is heavier in proportion to its surface.

Compare for instance a two inch and a three inch ball. The surface resistances are in the proportion of the

squares of the diameters; or as 4 is to 9. This is as 1 is to $2\frac{1}{4}$. The propelling force of inertia is in the ratio or proportion of the cubes, or as 8 is to 27; this is the ratio of 1 to $3\frac{3}{8}$. The larger shell with only $2\frac{1}{4}$ times the resistance of skin friction and cross-section of that of the larger shell has $3\frac{3}{8}$ times the propelling force of its inertia which as we know is in proportion to its weight. That is why a large projectile goes further than a small one.

This rule has many applications other than the one of projectiles. The force of gravity acts on all bodies in proportion to their weight; in a vacuum all bodies large or small fall with the same speed. In the air their fall is retarded by their skin friction and cross-sectional resistance as it may be called. Again the rule of the square and the cube comes into play and the larger body falls the faster. Dust falls with extreme slowness although it may be of the same material as a great rock which would fall with tremendous velocity.

The Four O'Clock Symbol

The clock faces, on which Roman numerals are used, it will be observed, have the four indicated by four I's, instead of by IV. It is said that the reason of this goes as far back as Charles the Fifth or as it is written Charles V. He is said to have declared that nothing should go before the V of his title, so he thus eliminated the usual IV, and there was nothing to do but to use four letter I's.

The Number 108 in Our Solar and Lunar Systems

The diameter of the sun is approximately 108 times that of the earth; the distance of the earth from the sun

is approximately 108 times the diameter of the sun. The distance from the moon to the earth is approximately 108 times the diameter of the moon.

Automobile Tires

For any size automobile wheel there is a specific tire, and besides this there is what is called an oversize tire. This is a tire of one half inch larger diameter of caliber than that of the regular size. The tire has to fit the same rim; the problem is what will be the increase in full diameter for the half inch increase in caliber.

As it has to fit the same rim the inner circle of the larger tire must be the same as that of the smaller one. The increased caliber adds a half inch all around the tire. Therefore as there is a half inch extra on every side of the tire it results in giving a tire one inch larger. Therefore the oversize for a 34×4 inch tire is $35 \times 4\frac{1}{2}$. Add an inch to the cross diameter of any given tire and a half inch to the caliber or as it may be termed to the small diameter and you will have the oversize.

If an automobile tire is priced at a certain amount the one of the next larger caliber, as a $4\frac{1}{2}$ and a 4 inch tire, seems always to be priced disproportionately high. But suppose we take a three inch tire to be compared with a three and a half inch one. If both were of identical thickness and of equally heavy construction the larger one would have $\frac{5}{30}$ ths extra material in its make-up. This is almost 17%, so on this basis alone it will be seen that there is a reason for a large increase in price. But an inch on the cross diameter although it sounds like more is much less; for two tires of 30 and 31 inch size

and of the same caliber the extra material in the larger one would be less than $3\frac{1}{2}\%$ of the total.

The Two Clerks

Two clerks start work in the same office at the same salary—each gets \$1,000 per annum. One has his pay increased \$50 every six months; the salary of the other is raised \$200 every year. The salaries are paid each half year. Which clerk fares the best?

At first sight it would seem that the clerk whose salary was raised \$200 each year would get the most, but it will be found on calculating the annual returns in each case that the clerk receiving the \$50 raise, every six months, keeps constantly ahead of the other. At the end of the first year the first clerk has received \$1,050 and the other one \$1,000; at the end of the second the first clerk will receive for his year of work \$1,250; the other one, \$1,200. It keeps on in the same way; the first clerk is always \$50 ahead of the second one.

Wine and Water Paradox

Let a tumbler, A , be half full of water, and a second tumbler, B , be half full of wine. A teaspoonful of wine is poured from B into A , and then a teaspoonful of the mixed water and wine is taken out of A and is put into B . The question is; after all this, has the tumbler, B , lost more wine than the water, which has been lost by the tumbler A ?

Most people will say at once that more wine was lost by B , than the water lost by A ; the correct answer is that they come out even, one losing less than a teaspoon-

ful of wine, the other the same quantity of water; each loses the same fraction of a spoonful.

Paradox of Squares of Fractions of a Number

A shepherd is to divide his flock in two unequal parts in such a way, that the square of the smaller part added to the larger part will be equal to the square of the larger part added to the smaller part. How does he do it?

This is an excellent catch and illustrates the difference between the powers of integral numbers and those of fractions. With integral numbers it is impossible, and the catch consists in doing it with fractions of the herd, dividing it into two fractions, each expressed as a fraction of the whole.

He may divide it into five eighths and three eighths, whose sum makes unity, and is taken as giving the whole flock. Performing the required operations, we have:

$$\left(\frac{5}{8}\right)^2 + \frac{3}{8} = \left(\frac{3}{8}\right)^2 + \frac{5}{8} = \frac{49}{64}$$

The calculation may be made with decimal fractions, thus:

$$(.625)^2 + .375 = (.375)^2 + .625 = .765625.$$

Divining Numbers Thought of

The following is an interesting example of old time arithmetical puzzles. It is taken from Laurechon, going back to the 17th century.

Let someone think of three numbers; he is to double the first one, add 5, and multiply the sum by 5, and add

10; to this he is to add the second number, and multiply by 10; then he is to add the third number, and subtract 350; the remainder will be the three numbers in their proper order, as he entered them into the computation.

Suppose the numbers thought of were 2, 3 and 4. We then proceed as follows: $(2 \times 2) + 5 = 9$; $9 \times 5 = 45$; $45 + 10 = 55$; $55 + 3 = 58$; $58 \times 10 = 580$; $580 + 4 = 584$; and finally, $584 - 350 = 234$, which are the three numbers thought of in the order in which they were entered into the computation.

A Paradox of the Time Card

The following is a good illustration of confusion between two units having the same name but different values.

A clerk applied for a job in a business house and said he was worth \$1,500 per annum. He was told he was not worth it.

Said the proprietor:

"There are 365 days in a year.....	365
You sleep 8 hours a day ($\frac{1}{3}$ of a day)....	122

Days left	243
You rest 8 hours a day.....	122

Days left	121
There are 52 Sundays in the year.....	52

Days left	69
You have one half day on Saturdays....	26

Days left	43
-----------------	----

You have $1\frac{1}{2}$ hours for lunch..... 28

Days left 15

You have two weeks' vacation..... 14

Days left 1

This is July 4th and we close, so that you do no work at all."

Remembering Telephone Numbers

It is generally unsafe to trust to the memory for too many telephone numbers. But there are in many cases peculiarities about them, which give a basis for retaining them in mind. Memoria Technica or Artificial Memory is the term describing such methods as we illustrate here.

Take the telephone number 2579. It is made up thus: the sum of the first two numbers, $2 + 5$, gives the third number, 7. The sum of the first and third number, $2 + 7$ gives the last number, 9.

Try the number 1140; the sum of the first two numbers gives half the third one.

Try 5292; the sum of the first, second and last number gives the third one.

Try 1121. The sum of the first two numbers gives the third number.

Try 190 and 187. 1 and 9 make 10, this gives the final cipher for 190; the sum of the first and last number gives the middle number of 187.

Try 6447. The sum of the two middle numbers gives 8; two greater than the first number and one greater than the last one.

Many numbers involve sequences or inverted sequences. Inversion may be partial or complete. Such are 4678 and 6876.

Making the Nine Digits in Proper Order Add up to One Hundred

It is required to combine the nine digits, keeping them in their proper order, so that they shall add up to 100.

The trick, for it is little better, is done by combining multiplication of the last two, i. e. 9 and 8, with the addition of the first seven digits, thus:

$$(9 \times 8) + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100$$

The same digits can be made to give the same sum in several other ways, none of any importance, as below:

$$15 + 36 + 47 = 98 + 2 = 100$$

$$(98 - 76) + 54 + 3 + 21 = 100$$

Curious Multiplication

The next series of operations brings out the nine digits in their natural and also in their inverted order. The calculation is self explanatory.

$$\begin{aligned} (1 \times 8) + 1 &= 9 \\ (12 \times 8) + 2 &= 98 \\ (123 \times 8) + 3 &= 987 \\ (1234 \times 8) + 4 &= 9876 \\ (12345 \times 8) + 5 &= 98765 \\ (123456 \times 8) + 6 &= 987654 \\ (1234567 \times 8) + 7 &= 9876543 \\ (12345678 \times 8) + 8 &= 98765432 \\ (123456789 \times 8) + 9 &= 987654321 \end{aligned}$$

A Unique Number

The smallest number, of which the alternate figures are ciphers, and which is divisible by 9 and also by 11 is the following:

9090909090909090909

Dividing it by 9 gives a series of tens; dividing it by 11 gives:

82644628099173553719

The quotient by 11 is quite curious; it contains various sequences of identical numbers in direct and reversed order; they are 8 2 6 4 and 4 6 2 8, 1 7 3 5 5 and 5 5 3 7 1 or better, instead of the last two, 1 7 3 5 and 5371. Others can be worked out.

Curious Multiplication and Addition

The following set of combined multiplication and addition is somewhat similar to the example already given.

$$\begin{aligned}
 (1 \times 9) + 2 &= 11 \\
 (12 \times 9) + 3 &= 111 \\
 (123 \times 9) + 4 &= 1111 \\
 (1234 \times 9) + 5 &= 11111 \\
 (12345 \times 9) + 6 &= 111111 \\
 (123456 \times 9) + 7 &= 1111111 \\
 (1234567 \times 9) + 8 &= 11111111 \\
 (12345678 \times 9) + 9 &= 111111111 \\
 (123456789 \times 9) + 10 &= 1111111111
 \end{aligned}$$

Multiplications of Nine

If we multiply 9 by 21 the product is 189; if the multiplication is by 321, the product is 2889. If successive

multiplications are carried out with the inverted digits, one being added to the row each time, the successive multipliers will be 21, 321, (as above) 4321, 54321, and so on up to the full row of digits inverted, 987654321. The left hand figures of the successive products will be the digits in regular order, the last figure in each case will be 9, and there will be 8's between the first and last digits, the number of such 8's being one less than the first or left hand figure of the multiplier. The successive products thus will be: 189, 2889, (as above) 38889, 488889, 5888889, 68888889, 788888889, 8888888889, 98888888889.

Sometimes to make it more striking a subtraction of 1 from each of the products is prescribed; this gives final 8's instead of 9's.

The successive multiplications may be arranged in a sort of pyramidal form, similar to the sets of multiplications above.

If the digits, omitting 8, are written down in their regular order, we shall have a number, which, if multiplied by 9, will give a succession of units or 1's. The multiplication follows:

$$\begin{array}{r}
 12345679 \\
 \times 9 \\
 \hline
 11111111
 \end{array}$$

One curious thing about this is the effect of omitting the 8; this omission gives the succession of 1's. If the 8 is not omitted, a cipher makes its appearance in the place below it. The multiplication follows:

$$\begin{array}{r} 123456789 \\ \times 9 \\ \hline 111111101 \end{array}$$

The reader may try placing the nine digits in reverse order and multiplying them by 9.

The following is a curious set of multiplications or of squarings:

$$\begin{aligned} II \times II &= I2I \\ III \times III &= I232I \\ IIII \times IIII &= I23432I \\ IIIII \times IIIII &= I2345432I \end{aligned}$$

The same operations can be carried out as far as desired, but the symmetry is impaired after the number of 1's exceeds nine.

Properties of Numbers

It is not easy to define the term "Properties of Numbers," but the conception of just what the term means is easily reached by examples. Thus one of the properties of the number one is that its square, cube and all other powers are equal each one to unity. Any number whose last two digits are divisible by four is divisible by four throughout. This may be taken as a property of the number four. A number is a perfect number if the sum of all its factors is equal to itself. Thus the factors of 6 are 1, 2 and 3, and the sum of these three figures is the original number 6. One of the properties of 6 then is that it is a perfect number. The factors of 28 are 1, 2, 4, 7, and 14; add these together and you will have 28. Therefore one of the properties of 28 is that it is a perfect number.

Properties of Nine

The number nine exceeds in interesting and practical properties all other digits. It is a wonder that it has not been more popular with the tribe of astrologers and others of that class, as a lucky or unlucky number, instead of three or seven. Even the superstitious crap player neglects it.

If 9 is used as the multiplier of any number from 1 to 20 the sum of the digits or figures of the product will be 9. Thus $13 \times 9 = 117$, and $1 + 1 + 7 = 9$. Or $18 \times 9 = 162$, and as before $1 + 6 + 2 = 9$.

Now write down the series of products of 9 by all the numbers from 1 to 20 and see if there is not something peculiar in this series of properties. Thus $9 \times 2 = 18$ and $9 \times 9 = 81$; each of these products is composed of the same digits but in reverse order. Try 9 multiplied by 13 and by 19; the products are respectively 117 and 171, the same digits but the last two are reversed. The numbers 10, 11 and 16 are the only multipliers which do not pair with any others as above, and if the cipher, 0, is left out of account, the refractory numbers reduce to 11 and 16; their products by 9 are respectively 99 and 144. A moment's inspection shows that they are incapable of reversing in the way all the other products are; hence they fall out by the wayside. Coming back to our paired numbers we will find that the sum of the multipliers for each pair under 10 times 9 will be the same, namely 11. Thus $2 \times 9 = 18$ pairs off with $9 \times 9 = 81$; $4 \times 9 = 36$ pairs off with $7 \times 9 = 63$. The two pairs of multipliers are 2 and 9 whose sum is 11, and 4 and 7 whose sum is also 11. This applies to single digit multipliers; now if we try the same with multipliers from

12 to 20, the sum of each pair of multipliers is 32. $13 \times 9 = 117$ and $19 \times 9 = 171$ are such a pair, and the sum of the multipliers is 32, thus $13 + 19 = 32$. After the multiplier 20 another pair comes in 21 and 22, and then another set from 23 to 30; this set and other succeeding sets may be left to the reader's investigations.

The Bookkeeper's Error

Suppose a bookkeeper finds a persistent difference in his trial balance. One of the properties of nine is that if any number is reversed or has its digits transposed, the difference between the two numbers will be divisible by 9 without a remainder. Thus transpose the digits of 279; this gives 972. The difference is 693, for $972 - 279 = 693$, and this difference is divisible by 9 without any remainder. If therefore the unhappy bookkeeper's difference is divisible by 9, it is almost certain that he has transposed some figures. This transposition of figures is a frequent source of trouble in bookkeeping, and the division by 9 without a remainder facilitates the finding of the error, when such error is due to transposition. Assume in the two columns of additions given below that the right hand one is correct, but that the left hand one is what the accountant really did. His erroneous inversion is put in italics.

6.35	6.35	173.21	173.21
1599.75	1599.57	187.26	178.26
381.23	381.23	—	—
—	—	360.47	351.47

1987.33 1987.15 and $1987.33 - 1987.15 = 18$;

and $1987.33 - 1987.15 = 18$; the divisibility of the

difference 18 by 9 without a remainder indicates the fact that two figures were transposed and makes the location of the error easier than it would be without this clue. The rule is not infallible; a difference divisible by 9 might exist and not be due to an inversion, but it is a case of probability; and inversion or transposition probably was the cause of the trouble. In the second pair the error is in the middle of the number but the same rule holds. It also holds for the transposition of several figures.

A Mystery in Money

Take any sum of money less than ten dollars (\$10.00) transpose the digits and take the difference between the two sums thus obtained. Then to the difference add its own transposition and you will always get the same amount, namely \$10.89. Try it and see. Here are a few examples.

	\$6.73	\$9.91	\$2.31	\$0.01
	3.76	1.99	1.32	1.00
dif.	2.97	7.92	0.99	0.99
	7.92	2.97	9.90	9.90
sum	\$10.89	\$10.89	\$10.89	\$10.89

Of course the dollars have nothing to do with it; they merely make it more picturesque and if it is used as a sort of trick or magic the dollar sign obscures it a little, a desirable feature in parlor magic and the like.

Thus a person is told to write down a sum of money less than ten dollars, and whose first and last figures must be different. Then he is to write down the trans-

posed figures directly under it. Ask him what the first or last figure is. As the sum of the first and last figures is always nine and the central figure is always nine you at once tell him what the number is. Then he is told to transpose this last number, writing it under the other, and to add the two. Then he is told what the result is, namely \$10.89. To make it a little more mysterious it is well to tell him to put any number you name below the lower pair of numbers and then to add them. All you have to do is to add this to 1,089 and tell him the result. The object of this is to prevent the repeated production of 1,089. The examples below illustrate this. In one case you can tell him to add 25, in the other case to add 31.

1.98	2.29
8.91	9.22
6.93	6.93
3.96	3.96
25	31
11.14	11.20

A professional magician as a rule avoids performing the same trick too often, so it is well to vary this one by sometimes not giving your friend the third number, but to let him finish the operation before telling the number found. It is well always to add a different number as just described, perhaps just once to omit the addition and telling him that he has ten dollars and eighty-nine cents as the result of his figuring. After that always add in some number and always a different one.

But while the dollars and cents are only a sort of concealing device in the above, it can be done with other denominate numbers so as to be much more puzzling. To do it with pounds, shillings and pence, the pounds must not exceed 12, and the pounds and pence must differ in number. The result will always be £12 18s. 11d. We will do it with £12 9s. 6d. and with £3 7s. 2d.

£12	9s.	6d.	£ 3	7s.	2d.
6	9	12	2	7	3
5	19	6	0	19	11
6	19	5	11	19	0
12	18	11	12	18	11

Divining a Sum of Digits

Another interesting bit of mystification may be based on properties of nine. Take any transposable number, that is to say any number whose first and last digits differ, and square it. Transpose it and square the result. Then on subtracting one from the other the result will be a number without any excess of nines. Thus a person is told to think of a number; then to square the number; then to transpose and square the transposed one; to subtract one from the other and to add the digits of the difference and to tell you the first or the last digit of this sum. At once you tell him the number which the addition of the digits has given. All you have to do is to annex to the number given a number which will make it a multiple of nine. If he says the first number is 1, the whole sum of the digits is 18; if 2 it is 27.

If you are asked to do it again introduce a slight variation. Let any two transposable pairs of numbers be multiplied together; then transpose one of them and multiply the original by the transposed one. Subtract one product from the other and again the result will be as before; the sum of the digits will be nine or a multiple thereof.

An example by the *first* method is given here.

$$\begin{array}{r} 3391 \times 3391 = 11498881 \\ 1933 \times 1933 = 3736489 \\ \hline \end{array}$$

Subtracting 7762392 whose digits added give 36 or 9×4

The *second* method may come next.

$$\begin{array}{r} 69 \times 69 = 4761 \\ 69 \times 96 = 6624 \\ \hline \end{array}$$

1863 whose digital sum is 18 or 9×2 .

When in the first case you are told that 3 is the first digit of the sum, you simply annex 6 to make 36; in the second case when told that 1 is the first number you annex 8 to make 18; this gives 36 and 18, both multiples of 9.

Such a number as 1,981 can be used because the two figures in the middle are reversible and carry out the law.

Other Mystifications

Take any number, invert it and subtract the smaller of the two from the larger; multiply by any number and

then cross out a number and give the number left; the digits are to be read in their order as a regular number. If this number is subtracted from the next highest multiple of nine, the difference between it and the next highest multiple of nine will be the number crossed out. Take the numbers 1,293, 146 and 97.

$$\begin{array}{r} 1293 \\ 3921 \\ \hline 2628 \end{array}$$

$$\begin{array}{r} 146 \\ 641 \\ \hline 495 \end{array}$$

$$\begin{array}{r} 97 \\ 79 \\ \hline 18 \end{array}$$

Cross out 6; 228 is left. The next highest multiple of 9 is 234, and $234 - 228 = 6$, the number crossed out.

Cross out 5; which leaves 49 — and we have $54 - 49 = 5$.

Cross out 8, leaving 10; this gives $18 - 10 = 8$.

If any of the differences had been multiplied it would have made no difference; it may be assumed that they are multiplied by 1. Thus multiply the first difference, 2,628 by 3; this gives 7884; cross out one of the 8's leaving 784; the next highest multiple of 9 is $9 \times 88 = 792$ and the difference is $792 - 784 = 8$, which was the number crossed out.

Take any two numbers less than 10; multiply one of them by 5, add 7, multiply by 2, add the other number, subtract 14; the number resulting will be made up of the digits of the two numbers.

Let the two numbers be 3 and 6. The successive operations are these: $3 \times 5 = 15$; $15 + 7 = 22$; 22×2

$=44$; $44 + 6 = 50$; $50 - 14 = 36$, which are the two digits we started with.

Next try the same operation with numbers of two digits, say 17 and 28. Carrying out the operations exactly as before gives as the final result, 198. This seems to be wrong. But if the middle digit is split up it can give $7 + 2 = 9$. We therefore read 198 as 17 and 28, which are the two numbers we started with. This can be done with all two digit numbers. The dividing the middle digit into two gives the original numbers.

The same sort of operation can be applied to three single numbers. Multiply the first number by 10 and add the second; multiply the sum by 10 and add the third number; the three numbers will then appear in their proper order.

Take the numbers 2, 4 and 7. Carrying out the operation gives $(2 \times 10) + 4 = 24$; $(24 \times 10) + 7 = 247$; the three numbers we started with in their proper order.

If this operation be followed out, it will be seen that nothing has been done except to multiply the first of the three numbers by 100, the second by 10 and the last is not multiplied by anything. These multiplications throw the three numbers into the hundreds, tens and units places in the final number.

The above principle can be applied to the investigation of the inversion operation just given. It gives a general formula for all cases. Call the three digits of the number to be treated a , b and c . Writing them as numbers, on the basis of the multiplications as above, gives $100a + 10b + c$. Writing this out again and placing beneath it its inversion, remembering that units now become hundreds and hundreds become units, and subtracting we have:

$$\begin{array}{r}
 100a + 10b + c \\
 100c + 10b + a \\
 \hline
 99a \qquad - 99c = 99(a - c)
 \end{array}$$

Suppose our number is 795; then $a - c$ is $7 - 5 = 2$, and $99 \times 2 = 198$ is the number which will be obtained by inverting and subtracting.

Dividing Multiples of Ten by Eleven

If 20 is divided by 11 the quotient is 1.81818. . . . If 30 is to be divided simply increase the first number of the last quotient by 1 and decrease the second number also by 1 and repeat the pair as long as you wish. The same rule applies all down the line, and the results of dividing multiples of 10 and 11 are tabulated here.

$20 \div 11 = 1.81818 \dots$	$60 \div 11 = 5.45454 \dots$
$30 \div 11 = 2.72727 \dots$	$70 \div 11 = 6.36363 \dots$
$40 \div 11 = 3.63636 \dots$	$80 \div 11 = 7.27272 \dots$
$50 \div 11 = 4.54545 \dots$	$90 \div 11 = 8.18181 \dots$

Inspection will show perfectly how the rule of adding and subtracting 1 is carried out. Another thing to be observed is that each pair or couple of digits in the quotients above is divisible by 9; it makes no difference what pair of numbers are taken as long as they are next to one another in any of the quotients. From the first quotient of the tabulation we get the pairs 18 and 81, both divisible by 9, and the same applies to all the other quotients.

A MANUAL OF THE SLIDE RULE

ITS HISTORY, PRINCIPLE AND OPERATION

BY

J. E. THOMPSON, B.S. IN E.E., A.M.

*Department of Mathematics, Pratt Institute,
Brooklyn, N. Y.*

Copyright, 1930
By D. VAN NOSTRAND COMPANY, INC.

All Rights Reserved

*This book or any part thereof may not
be reproduced in any form without
written permission from the publishers.*

PRINTED IN THE U. S.,

PREFACE

Accounts of the slide rule are usually confined to encyclopedia articles, supplementary chapters of books on other subjects, and manufacturers' booklets of directions for its use. The first two are usually sketchy and incomplete and intended for general information rather than for specific instruction. The manufacturers' booklets are well suited to their purpose, which is to give explicit directions for the mechanical operation of some one form of the rule and examples illustrating its use in particular calculations, but are not intended as general books on the slide rule.

The present manual is intended to supply in a uniform presentation an account of the history, principles and practical uses of the slide rule. Among its specific aims are: to show the unity of the principle of operation of all slide rules, to explain this principle in a manner which shall be easily understood, to give in clear and simple language explicit and inclusive instructions for the use of all the standard forms, and to supply a fairly complete account of the history of the slide rule and a description of the more usual of the special forms of the rule. The material in Chapter IV is intended to be more suggestive than exhaustive and that in Chapter V is intended more as supplementary general information than as instruction in the use of those rules described.

It is hoped that Chapters II, III and IV will be of value to all users of the slide rule and suitable for self instruction,

and that Chapters I and V will be of interest to all who pay any attention to the slide rule in any way. In the preparation of Chapters II, III and IV, great care has been used to make the descriptions and explanations complete, simple and clear. The style adopted has been found very useful and effective in the author's teaching experience with users of the slide rule, among whom have been numbered high school and college students, practical industrial workers and clerical workers. It is for these classes of readers that the book is designed.

It is a pleasure to acknowledge a great and peculiar obligation to Mr. W. H. H. Cowles of Pratt Institute. He originally undertook the preparation of the book and had prepared a tentative list of chapter headings and outlined the scope of the book when new duties attendant upon a change in his relation to the Institute necessitated his relinquishing the work. The present writer undertook the work at Mr. Cowles' suggestion and has since profited by frequent discussions with him.

The kindness and courtesy of Keuffel & Esser Company has made possible the inclusion of almost all the material of Chapters IV and V, and the plates for a number of the illustrations were graciously loaned by them. To this company the obligations of the author and publisher are hereby acknowledged with thanks.

As the proof has been read by the author alone errors will undoubtedly be found. Notices of these will be gladly received by him or the publisher.

BROOKLYN, N. Y.

J. E. THOMPSON.

September, 1930

CONTENTS

CHAPTER I

HISTORY OF THE SLIDE RULE

ARTICLE	PAGE
1. Introduction.—Logarithms	1
2. Logarithmic Multiplication and Division and the Invention of the Slide Rule	3
3. Improvement and Development of the Slide Rule in England	6
4. The Slide Rule in France	9
5. The Mannheim Slide Rule	13
6. Recent Developments	15
7. The Slide Rule in the United States	16

CHAPTER II

THEORY AND OPERATION OF THE MANNHEIM SLIDE RULE

A. THEORY OF THE RULE

8. Some Properties of Exponents	21
9. Logarithms	23
10. Principle of the Slide Rule	25
11. Application of the Slide Rule Principle	30
12. The Standard Mannheim Slide Rule	33

B. OPERATION OF THE RULE

13. Division and Reading of the Scales	35
14. Multiplication with the Slide Rule	40
15. Division and Reciprocals	48
16. The Decimal Point in Slide Rule Multiplication and Division	53

ARTICLE	PAGE
17. Combined Multiplication and Division	58
18. Proportion	62
19a. Squares, Cubes and Roots	63
19b. Use of Gauge Points on the A and B Scales	73
20. The Inverted Slide	74
21. The Scale of Equal Parts (L) and Its Use	78
22. The Sine Scale (S) and Its Use	80
23. The Tangent Scale (T) and Its Use	84

CHAPTER III

MODIFIED FORMS OF THE MANNHEIM RULE AND THEIR USE

A. THE POLYPHASE MANNHEIM SLIDE RULE

24. The Polyphase Mannheim Slide Rule	88
25. The Inverted C Scale and Its Use	90
26. The Cube Scale and Its Use	95

B. THE POLYPHASE DUPLEX SLIDE RULE

27. The Duplex Slide Rule and the Folded Scales	99
28. Operation of the Polyphase Duplex Slide Rule	102

C. THE LOG-LOG DUPLEX SLIDE RULE

29. Natural Logarithms and Exponentials	109
30. Exponentials, Powers and Roots and the Slide Rule	115
31. The Log-log Scale and the Log-log Duplex Slide Rule	117
32. Use of the Log-log Scales	121

D. THE SLIDE RULE IN TRIGONOMETRY

33. Introductory	133
34. Radian and Degree Angle Measure Conversion	134
35. Solution of Right Triangles	135
36. Solution of Oblique Triangles	140
37. Slide Rule Check of Logarithmic Solutions	146
38. Interpolation of Logarithms	147

CHAPTER IV

TYPICAL PROBLEMS AND SLIDE RULE SETTINGS

	PAGE
A. INTRODUCTION	151
B. TYPICAL PROBLEMS AND SETTINGS FOR A, B, C, AND D SCALES	152
C. TABLES OF CONVERSIONS AND GAUGE POINTS FOR C AND D SCALES :	164
D. SETTINGS FOR POLYPHASE MANNHEIM RULE .	171
E. SETTINGS FOR POLYPHASE DUPLEX AND LOG-LOG DUPLEX RULES	177
F. SETTINGS AND TYPICAL PROBLEMS INVOLVING THE LOG-LOG SCALES	185

CHAPTER V

SPECIAL FORMS OF THE SLIDE RULE

ARTICLE

39. Introduction	204
40. Slide Rules of Mannheim Form with Special Scales] .	206
41. Duplex Slide Rules with Special Scales	210
42. Circular Slide Rules	214
43. Cylindrical Slide Rules	217

CHAPTER I

HISTORY OF THE SLIDE RULE

1. Introduction — Logarithms. — The modern slide rule is frequently thought of as a modern invention but in its earliest forms it is several hundred years old. As a matter of fact, it can hardly be called an invention in the ordinary sense, but is rather an outgrowth of certain ideas in mathematics. In this chapter we shall trace briefly the origin of the slide rule in these ideas and the development of its various forms into the familiar instrument of today.

Insofar as the slide rule can be called a distinct invention it originated as a scale on which were laid off lengths representing the logarithms of numbers so that multiplication and division could be performed mechanically by finding the sums and differences of lengths on this scale. But such a scale is simply a condensed table of logarithms, so the history of the slide rule in reality begins with the invention of logarithms.

Logarithms were invented by John Napier, Baron of Merchiston in Scotland, and his discovery was first

publicly announced in 1614. In this announcement Napier stated his purpose in the following words: "Seeing there is nothing (right well beloved Students of Mathematics) that is so troublesome to mathematical practice; nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which beside the tedious expense of time are for the most part subject to slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances." The slide rule represents the most "certain and ready" fulfillment of this purpose, and is the final result of his labors. Although the completed invention of logarithms was not publicly announced until 1614, Napier had privately communicated a summary of his results to the Danish astronomer, Tycho Brahe, in 1594. Thus in its fundamental principle the slide rule dates back nearly three and one half centuries, beginning only a hundred years after the discovery of America.

In his book of 1614 on logarithms Napier explains their nature by a comparison between corresponding terms of an arithmetical and a geometrical progression. He gives tables of the logarithms of the sines and tangents of angles calculated to seven decimal places. His definition of the logarithm of a number N was what we should express, according to our present definition, by the

quantity $10^7 \log_e \left(\frac{10^7}{N} \right)$, where $e = 2.71828$ approximately.

The transformation to the form described in the present day definition was made by Henry Briggs, Professor of Mathematics at Oxford University, who in 1617 brought out the first table of ordinary logarithms of the natural numbers from 1 to 1000 to the base 10 calculated to fourteen decimal places. These were later extended and now tables of all the numbers from 1 to 108000 are common and some have been computed to sixty decimal places.

2. Logarithmic Multiplication and Division and the Invention of the Slide Rule.—In order to carry out the multiplication of numbers we add their logarithms and take the “antilog” of the sum; thus if $x = AB$ then $\log x = (\log A + \log B)$ and the product $x = \text{antilog} (\log A + \log B)$, to any base whatever. Similarly if $y = \frac{A}{B}$, $y = \text{antilog} (\log A - \log B)$. The logarithms of A and B are found in a table of logarithms and the antilogarithm of the sum or difference is found in the same table. This process may also easily be carried out mechanically in the following manner: Lay off to a convenient scale on a straight line the logarithms of the natural numbers and instead of marking $\log 1$, $\log 2$, $\log 3$, etc., on the scale simply mark 1, 2, 3, etc. Then to find $x = AB$ measure by a pair of dividers the scale length of $\log B$,

that is, the length from the beginning of the scale to the point marked B . Now lay off on the scale with the dividers this same length beginning at the point marked A . The total length is equal to $(\log A + \log B)$ and the number at the end of this total length on the scale is x . Such a logarithm scale, laid off on a strip of wood, was first constructed by Edmund Gunter, Professor of Astronomy at Gresham College, in London, in 1620. This device was known as Gunter's Scale or Gunter's Rule. Obviously Gunter's Rule may be used for division by measuring the difference between the scale lengths for dividend and divisor, that is, $\log A - \log B$.

The process just described can be much more easily and quickly carried out if we use two similar scales and instead of measuring and adding or subtracting the lengths on one scale, simply lay one upon or beside the other, with the beginning of the second scale placed at the point corresponding to A on the first. Then the point corresponding to B on the second will be beside the point on the first which is at the end of a length equal to the sum of the two lengths, that is, the point corresponding to $x = AB$. Similarly the length equal to $\log A - \log B$ may be found and the point at the end of this length will be $y = \frac{A}{B}$. Such a combination of Gunter's Rules was first made by Rev. William Oughtred

who lived near London, and was first made public by William Forster, a teacher of mathematics in London, in 1632, when he stated that he first saw the double rule at Oughtred's home in 1630 at which time Oughtred told him he had constructed the double rule several years previously. Oughtred's double rule was made in the straight form with one rule to be held against and *slide* along the other by hand, and also in circular form on cardboard, one scale being on the perimeter of a circular disc which was pivoted at the center of a similar and larger disc and turned upon it, so that the smaller circular scale lay just inside of and in alignment with the larger, which was laid off on the perimeter of the larger disc. These rules carried scales of the logarithms of the natural and trigonometric numbers. William Oughtred is thus the inventor of the *slide rule* in both the straight and circular forms.

The rules were for the first time made up of three strips of wood with the two outer strips held together by bridging cleats and the third strip sliding freely between them edge to edge, by Seth Partridge, a surveyor and teacher of mathematics, in 1657. This combination rule carried the number and trigonometric scales on both faces and was thus the first complete slide rule. It was as a matter of fact a complete *duplex slide rule*, which is thus now about 270 years old.

3. Improvement and Development of the Slide Rule in England. — The name “sliding-rule” (later abbreviated to “slide rule”) was first used by Thomas Everard, a mechanic and excise officer at Southampton, who in 1683 brought out rules adapted for the computations involved in determining cubic contents of measuring vessels and containers of market commodities. On his rules were marked so-called “gauge points” or numbers often used in slide rule settings involving the standard measurements and measuring unit conversion factors. These were the first gaugers’ or commercial slide rules. On these rules appeared for the first time scales which could be used for finding the square and cube roots of numbers. The first slide rule scales designed especially for the direct reading of the squares and cubes and square and cube roots of numbers were made by one John Warner in the year 1722.

In the year 1697 William Hunt, of whom very little is known, brought out a slide rule having in addition to the usual scales a scale giving the areas of circles of known diameters, one for finding the perimeters of ellipses of known axes, and a scale for finding the length or width of a rectangle of unit area when the width or length respectively was known. Since when the area is unity, $1, WL = 1$ and hence $L = \frac{1}{W}$ or $W = \frac{1}{L}$, this scale was

in reality a scale of *reciprocals*, now known as the "inverted scale." The reciprocal or inverted scale was first placed on the rule as a simple regular scale running in the reverse direction by William Hyde Wollaston soon after the year 1797. Incidentally, this same Wollaston brought out in 1815 a slide rule with scales specially adapted for the calculations involved in chemistry.

A development or use of the slide rule which is not commonly known but which later led to a very important improvement was made by Sir Isaac Newton, the inventor of the infinitesimal calculus, discoverer of the law of gravitation, and founder of modern mechanics and dynamical astronomy. In his work in the theory of algebraic equations he devised a method of solving cubic (third degree) equations in which three movable slide rule scales were laid side by side and a separate straight edge laid across them to bring together or in line three certain numbers on scales not contiguous to one another. This is the first recorded use of what is now known as a *runner*. In discussing Newton's work this device was made more definite and described as a "marking line moving parallel to itself" by E. Stone in 1726. This moving marking line was first definitely attached to the slide rule in a regular manner as a mechanical part of the rule itself in 1775 by John Robertson, teacher of mathematics at the Royal Academy at Portsmouth and later Librarian of the Royal Society of London.

The circular slide rule invented in England by Oughtred about or before 1630, as stated above, was independently re-invented in France in 1716 by Jean Baptiste Clairaut, the father of the famous mathematician of that name, but was used hardly at all until 1748. In that year it was brought out in a more complete and convenient form by George Adams, manufacturer of mechanical instruments for King George III. Circular and spiral forms of the scales were much improved toward the end of the 18th century by William Nicholson (see below).

Before 1775 the various slide rules in use in England and the American colonies were very poorly constructed and ruled. In the year 1707 one John Ward in a work on practical mathematics and gauging wrote that results obtained by their use were so inaccurate that the pen should always be used in such work, though he later stated that he did not refer to the long rules. About 1775 James Watt, one of the inventors of the steam engine, had made to his own order and with great care excellent rules for his own use in engine design. This seems to have had the double effect of setting a high standard of workmanship and popularizing the use of slide rules among English engineers.

About or soon after this time William Nicholson (born 1753, died 1815) publisher and editor of "Nicholson's Journal," a kind of technical journal, began to devote

most of his time to the study and improvement of the slide rule and used his "Journal" to advertise various rules and promote their use. He especially improved the circular and spiral forms of rules. In "Nicholson's Journal" there was announced in 1817 a new development which, while not used for a long time, was an important development; it was due to Sylvanus Bevan. This was the introduction of what is now known as the *folded scale*. This consisted of a scale of the same length and graduations as the usual logarithmic scale but divided into two parts so that it began at the left-hand end of the rule at a point a little past 3 and proceeded to 10 near the center, and beginning again at that point with 1 proceeded to the same number at the right-hand end as was placed at the left. Thus the normal scale was "folded" at the middle so that the ends were together; it was then parted at the folding point and these separated section ends were used as the ends of the new scale.

After Nicholson's day the slide rule was neglected and fell into disuse in England. France became the leading country in its development and production.

4. The Slide Rule in France. — Gunter's scale was introduced into France in 1624 by Edmund Wingate, an early and prolific English writer on the slide rule, who was for a long time believed by many to have been

the inventor of the "sliding-rule." It has been definitely shown, however, that he merely added a scale of equal parts to Gunter's Rule so as to show the logarithms of the numbers on Gunter's scale, and that he used the slide rule only after it was made public by Oughtred.

For a long time the French did little in the way of development or improvement and merely used the English rules first advertised and taught by Wingate; even these were not much used or known before about 1700. In fact, this was the case all over the continent of Europe, and the first mention of the slide rule by a German writer is in a Latin work on mathematical instruments written by one Biler published in 1696. Throughout Europe, slide rules were not popular during the 17th and 18th centuries but, as we have already seen, they were popular in England, where decimal fractions and logarithms were much more a part of elementary mathematical education than on the continent.

The time of the French Revolution and the Napoleonic Wars, however, was a period of intense intellectual activity in France. Some of the world's greatest mathematicians and mathematical physicists (Lagrange, Laplace, Legendre, John Bernoulli the younger, Fourier, Poisson, Ampère, etc.) flourished there in that period. Besides the great development in mathematical analysis there were introduced a new calendar and a new system of

angle measurement, though these were never widely used, the decimal (so-called "metric") system of measurement was invented and established, new and wonderful mathematical tables and methods of computation were produced, and the slide rule came into use and was improved in design and brought to a high state of mechanical perfection. Toward the middle of the century just dawning France became the center of activity in the study and production of the slide rule and the government made knowledge of the slide rule a requirement for admission into all technical public service.

Other European countries still lagged behind, however, although in 1843 an Austrian, L. G. Schulze von Strasnitzki, Professor at the Royal Institute at Vienna, made and used a slide rule eight feet long for demonstration and instruction purposes. Such a rule was at one time soon afterward used by the firm of Tavernier-Gravet in Paris and is now supplied in the United States by Keuffel & Esser.

In 1716, as we have seen, the circular rule was independently re-invented in France and circular rules have been popular there for a long time, but beyond the general mechanical improvement and development the greatest distinct contribution to the slide rule was the *log-log* scale, which was invented in 1815 by Peter M. Roget, a French physician, and reported by him in that year to the Royal

Society of London. Roget called his scale a "logometric" scale and pointed out that it could be used for raising any number to not only the second or third but to any power (within any desired range determined by the length of the scale) integral or fractional, and for taking corresponding roots. He called attention to its use in many calculations involving exponentials (population increase, compound interest, probability, theory of logarithms, etc.) but such computations were not then of much practical use and the log-log scale fell into disuse and was forgotten. It was later independently re-invented in France by Burdon, who proposed arrangements for finding x and y in such simultaneous combinations as $xy^m = a$, $xy^n = b$. The case of $x = \sqrt[n]{a}$ when a is a fraction was first provided for by F. Blanc in Germany before 1900. Due to the requirements of calculations in thermodynamics, hydraulics, electricity, etc., the various forms of log-log scales were independently re-invented in England by several, including Professor John Perry, the famous engineering mathematician and teacher. One of them, Professor C. S. Jackson, says, "The use of a log-log scale for numbers less than unity was one fondly thought new, but in this idea we were all anticipated by Blanc," and of course in the original idea of the log-log scale Blanc, as well as Burdon and the others, was long anticipated by Roget.

Log-log scales designed for the solution of algebraic equations up to and including the quintic, carrying out Newton's idea, were introduced by Furler in Germany in 1899 and scales for use with complex quantities $a + b\sqrt{-1}$ were designed by R. Mehmke in Germany soon after 1900.

The next distinct advance after the invention of the log-log scale in 1815 was made by Lenoir in France in 1821. He constructed after designs of Jamard and Collardean a machine for dividing the scales, which work had previously been done by hand. He developed this dividing engine so that it would mark the scale divisions simultaneously upon eight rules 25 centimeters long, executing very fine work.

The most important step in the evolution of the slide rule after Roget's invention was also made in France, in 1850, and is due to Lieutenant Amédée Mannheim. This development has since proved to be of such importance that it will be considered separately in the next article. The rule itself will be described in detail in a later chapter.

5. The Mannheim Slide Rule.—Amédée Mannheim, a Frenchman with a German name, was born in 1831 and died in 1906 after a long and illustrious career as an artillery officer and as Professor of Mathematics at L'École Polytechnique in Paris. It was in 1850 when

only about nineteen years old that he designed his slide rule. He was then an artillery officer at Metz, one of the great fortresses of Alsace-Lorraine, where he had gone as Lieutenant of Artillery after he entered L'École Polytechnique in 1848. Besides his military contributions and his slide rule he has contributed largely by his researches to geometry and mechanics, and as Colonel Mannheim homage was rendered him at L'École Polytechnique on the occasion of the celebration of his seventieth birthday in 1901.

Lieutenant Mannheim designed his slide rule in 1850 but it was not publicly announced until December, 1851, when he described the rule and its use in a publication entitled "Modified Calculating Rule.—Instructions." The Mannheim Rule was adopted by the French Artillery and manufactured by the famous firm of Tavernier-Gravet. It was several years, however, before it became known in other countries and it did not come into general use until a book on the slide rule which highly recommended the Mannheim Rule was published in 1859 by an Italian named Q. Sella. This book made such an impression that the rule was soon in general use in Europe and the year 1859 is sometimes erroneously given as the year of its invention. The Mannheim Rule was hardly known in England for twenty years after its invention and was very little used in the United States before 1890.

6. Recent Developments. — “Great activity has been shown in recent times in the direction of perfecting the mechanical execution of slide rules. In this, Germany took the lead. After the slide rules used in Germany had been imported from France for a period of 20 or 30 years German manufacturers gained the ascendancy. In 1886 Dennert & Pape in Altona began to mount the scales of numbers upon white celluloid instead of boxwood or, as in some cases, metal. This method has now become well nigh universal. In America celluloid scale rules were being sold by Keuffel & Esser as early as 1886 and since 1886 this firm has manufactured them in Hoboken, N. J. A prominent German firm engaged in the manufacture of slide rules is that of Albert Nestler in Baden. The firm of A. W. Faber has factories in Germany and branch houses in England, France and the United States.

“Slide rule scales are now usually engine divided and have reached a very high state of perfection in both workmanship and mathematical accuracy.

“For the purpose of securing greater accuracy in the use of the slide rule without increasing the length of the divisions, to secure, say, the fourth [or fifth] figure with a 25-centimeter rule, attachments have been made which [are either magnifying glass runners or verniers and frequently] go under the name of “cursors.” An early attempt along this line was a vernier applied by J. F.

Artur in Paris. Similar suggestions were made by O. Seyffert in Germany. Perhaps the best known of these is the Goulding Cursor, which allows the space between two consecutive smallest divisions to be divided into ten parts."¹ Probably the most convenient magnifying runner is that supplied by Keuffel & Esser, although very similar runners are supplied by other manufacturers. This is mounted in a metal frame and is applied to the rule by fastening it to the regular runner with a spring clamp. This firm also supplies a so-called "decimal indicator" which is a small semicircular scale attached as an integral part of the runner, and bearing divisions indicated by a pointer as subdivisions of the main scale.

Many special forms of slide rule have been developed for particular uses; some of these will be described in a later chapter. In his "History of the Slide Rule" Cajori lists 256 slide rules which in 1909 had appeared since 1800 and says that the list is probably not at all complete.

7. The Slide Rule in the United States. — Before about 1880 the slide rule was but little known and very little used in the British American Colonies and the United States and references to it in American technical and mathematical literature were few and sketchy. An account of the rule which probably reached the engineering

¹ From Cajori's "History of the Slide Rule."

profession more widely than any other was published in New York in 1856 in the "Mechanics and Engineers Reference Book." This book was written by Charles Haslett, Civil Engineer, and Charles W. Hackley, Professor of Mathematics at Columbia University (then Columbia College), and devotes five pages to the slide rule.

One of the earliest slide rules produced in America was Palmer's "Computing Scale" which appeared in Boston in 1844. It was a circular rule eight inches in diameter, thus providing a scale about twenty-five inches long. It was later modified and carried all the usual scales of that time but seems not to have been used outside of Massachusetts and New York.

The slide rule first began to come into real use in the United States soon after 1880 when the Mannheim Rule was first introduced. About that time it was made a part of the required courses in mathematics and engineering at Washington University in St. Louis, an institution famous for its early work in mathematics, astronomy, physics and engineering and for long the scene of the labors of William Chauvenet, the celebrated American mathematician, astronomer and teacher. In 1881 the well known Thacher Cylindrical Slide Rule was patented by Edwin Thacher, a graduate of Rensselaer Polytechnic Institute and a bridge engineer. This rule will be de-

scribed in a later chapter. Following this invention, the work at Washington University, and the introduction of the Mannheim Rule, the use of the slide rule began to spread but the rule was not generally popular until William Cox began his work in 1890.

In that year Cox began a long campaign of propaganda and education in the columns of the "Engineering News," published in New York. He took the Mannheim Slide Rule as standard and wrote articles describing the rule and its use and preaching its value and importance to engineers and other technical workers. He patented several modifications of the Mannheim Rule which were produced by the firm of Keuffel & Esser and in 1891 wrote a manual of instruction for users of the slide rule, which was also published by Keuffel & Esser and which, with revisions, remained in wide use for thirty-five years.

In 1891 Cox patented his duplex slide rule. In this rule there are three strips of equal thickness, the slide and two outer pieces, these outer pieces being rigidly held together by metal cleats and grooved on the inner edges so that the slide fits snugly and slides freely between them with the faces of all three strips flush on both sides. This rule is thus mechanically the same as the Seth Partridge rule of 1657. The novelty and advantage of the Cox Duplex rule lay in its use of the Mannheim scales and the printing of scales on both sides of the

entire rule. The inverted, equal parts, cube and folded scales were, about 1897, added to the Cox Duplex rule and the resulting rule called the "Polyphase Duplex" rule. This rule has now superseded the original Cox duplex and is widely used among engineers.

The standard Mannheim Rule carries scales A and D on the stock or body of the rule and scales B and C on the face of the slide, with scales of sines, tangents and equal parts on the back of the slide. In 1900 Keuffel & Esser added an inverted C scale to the face of the slide and a scale of cubes of C scale numbers to the stock. This modification of the Mannheim Rule is called the "Polyphase Mannheim" rule. In the same year this same firm also brought out an improvement in the mechanical construction of the Mannheim Rule. This consists in making one side of the groove in which the slide fits, movable on the main body of the stock. With this improvement the pressure of the sides of the groove on the slide can be adjusted to supply the proper friction for the best operation of the slide. The duplex rules of Keuffel & Esser are also made adjustable by making one of the strips of the stock, between which the slide moves, moveable perpendicularly to its length between the metal cleats which hold them.

The most complete of the standard form rules, also supplied by Keuffel & Esser, is their "Log-log Duplex"

rule, brought out in 1908. It is constructed exactly like the Polyphase Duplex with the exception that it is wider, and in addition to all the scales of that rule carries a log-log scale in four parts, each running the full length of the rule. One gives $x = e^n$ when x is between 0 and 1 (that is, a fraction) and the other three give $x = e^n$ when x is between 1 and 22000. This improvement adds greatly to the value of the rule in connection with the calculations of modern physics and engineering.

Keuffel & Esser supply a number of other types of slide rule for special purposes, some of which will be described in a later chapter. Besides this firm whose series is perhaps the most complete, there is another American firm which produces excellent slide rules; this is Eugene Dietzgen Company. Their rules are of the forms now standard in the United States and a large part of the world, but those rules other than the standard Mannheim Rule bear trade names specially adopted by the producer. Later descriptions of standard forms of slide rules in this book will apply equally well to the rules of all makers who supply such rules. The types of rules described in this book will be those most used in the United States and the names used in cases where the rule is supplied by only one maker will be those adopted by that maker.

CHAPTER II

THEORY AND OPERATION OF THE MANNHEIM SLIDE RULE

A. THEORY OF THE RULE.

8. **Some Properties of Exponents.** — We give here for reference and review a brief summary of some of the chief properties of algebraic exponents. Their demonstrations can be found in any good text book of algebra and the statements given here are to be considered as more descriptive than demonstrative.

If a , b , m , n are any real numbers (that is, numbers not involving $\sqrt{-1}$), then a^m represents the continued product of a taken m times as a factor and

$$a^m \times a^n = a^{m+n} \quad (1)$$

represents the product of a used $m + n$ times as a factor. Similarly

$$a^m \div a^n = a^{m-n} \quad (2)$$

indicates the use of a as a multiplier m times followed by its use as a divisor n times so that the net result is the use of a as a factor $m - n$ times. In the same way as in (1) if a is used m times as a factor and the result

so obtained is itself in turn used n times, then a will be used altogether $m \times n$ times. Therefore

$$(a^m)^n = a^{mn}. \quad (3)$$

If in (2) we make $n = m$ we get

$$a^m \div a^m = a^{m-m} = a^0;$$

but $a^m \div a^m = 1$ and therefore

$$a^0 = 1. \quad (4)$$

Therefore if in (2) we put $m = 0$ we get $a^0 \div a^n = a^{0-n} = a^{-n}$; but $a^0 \div a^n = 1 \div a^n$. Therefore

$$a^{-n} = \frac{1}{a^n}. \quad (5)$$

Similarly
$$\frac{1}{a^{-n}} = a^n. \quad (6)$$

By definition the square root of a number is that number which multiplied by itself produces the given number. Thus $\sqrt{a} \times \sqrt{a} = a$. Similarly $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$, for the cube root, and for the n th root $\sqrt[n]{a} \times \sqrt[n]{a} \times \sqrt[n]{a} \times \dots$ n times $= a$. That is, $\sqrt[n]{a}$ raised to the n th power gives a , or $(\sqrt[n]{a})^n = a$. But we have also by (3), $(a^{1/n})^n = a^{1/n \times n} = a^1 = a$. Hence

$$a^{1/n} = \sqrt[n]{a}. \quad (7)$$

Again by (3), $(a^{1/n})^m = a^{(1/n \times m)} = a^{\frac{m}{n}}. \quad (8)$

Also, we get the same result if we raise a to the m th power and then take the n th root of the quantity so obtained or if we first take the n th root of a and then

raise this quantity to the m th power; that is $\sqrt[n]{(a^m)} = (\sqrt[n]{a})^m$. But by (7) $\sqrt[n]{a} = a^{1/n}$, and hence according to (8), $(\sqrt[n]{a})^m = (a^{1/n})^m = a^{m/n}$. Thus we have

$$a^{m/n} = \sqrt[n]{(a^m)}. \quad (9)$$

Now by definition $(ab)^2 = (ab) \times (ab) = (aa) \times (bb) = a^2b^2$; $(ab)^3 = (ab) \times (ab) \times (ab) = (aaa) \times (bbb) = a^3b^3$; and so on for any other exponent. Thus we can write

$$(ab)^m = a^mb^m. \quad (10)$$

From this result we can write $(ab)^{1/n} = a^{1/n}b^{1/n}$, or by (7)

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}. \quad (11)$$

In the same way $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \cdot \frac{a}{b} = \frac{aa}{bb} = \frac{a^2}{b^2}$, and so on; thus

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (12)$$

and

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}. \quad (13)$$

(It is a common error for students to write, analogously to (12) and (13), $\sqrt{(a+b)} = \sqrt{a} + \sqrt{b}$, $\sqrt{(a-b)} = \sqrt{a} - \sqrt{b}$, and similarly for other roots and powers, but that this is not true is easily seen by taking numerical values for a and b . Therefore we can NOT write $\sqrt[n]{a \pm b} = \sqrt[n]{a} \pm \sqrt[n]{b}$, $(a \pm b)^n = a^n \pm b^n$.)

9. Logarithms. — Based on the formulas of Article 8 we give here in the same summary manner some of the

chief elementary properties of logarithms. By definition "the logarithm of any number A to the base b is the exponent m of the power to which b must be raised in order to produce A ." Thus if

$$\left. \begin{array}{l} m = \log_b A \\ b^m = A. \end{array} \right\} \quad (14)$$

then

and a logarithm obeys all the rules of exponents. Thus if $m = \log_b A$ and $n = \log_b B$, then by the second of (14) and by (1) $AB = b^m \times b^n = b^{(m+n)}$ and since now $b^{(m+n)} = AB$, then by (14) $\log(AB) = m + n$. That is,

$$\log(AB) = \log A + \log B. \quad (15)$$

In a similar manner from (2) and (14) we get

$$\log\left(\frac{A}{B}\right) = \log A - \log B. \quad (16)$$

From (15) $\log(A^2) = \log(AA) = \log A + \log A$, that is $\log(A^2) = 2(\log A)$.

Similarly $\log(A^3) = \log(AAA) = \log A + \log A + \log A = 3(\log A)$, and so on for any exponent. Thus

$$\log(A^m) = m \log A. \quad (17)$$

If instead of the exponent m we use the exponent $\frac{1}{n}$ this equation gives $\log(A^{1/n}) = \frac{1}{n}(\log A)$. But by (7) $A^{1/n} = \sqrt[n]{A}$. Therefore

$$\log(\sqrt[n]{A}) = \frac{\log A}{n}. \quad (18)$$

If now we use (17), $\log[(AB)^m] = m \log(AB)$.

Hence by (15)

$$\log[(AB)^m] = m(\log A + \log B). \quad (19)$$

Similarly using (18) and (15),

$$\log[\sqrt[n]{AB}] = \frac{1}{n}(\log A + \log B); \quad (20)$$

using (17) and (16),

$$\log\left[\left(\frac{A}{B}\right)^m\right] = m(\log A - \log B); \quad (21)$$

using (18) and (16),

$$\log\left[\sqrt[n]{\frac{A}{B}}\right] = \frac{1}{n}(\log A - \log B), \quad (22)$$

and so on for any combination of the formulas (1) to (13).

In particular we will note that since by (4) $b^0 = 1$, then by (14) to any base whatever

$$\left. \begin{array}{l} \log 1 = 0 \\ \text{and since } b^1 = b, \\ \log_b b = 1. \end{array} \right\} \quad (23)$$

Certain of the formulas in this article are fundamental in the theory of the slide rule; all are useful toward an understanding of its operation.

10. Principle of the Slide Rule. — The slide rule is a device by means of which multiplication, division, raising to powers, root extraction and their combinations may quickly and easily be performed by carrying out me-

chanically the logarithmic operations indicated by formulas (15) to (18) and the related formulas. This is readily understood by reference to the following diagrams.

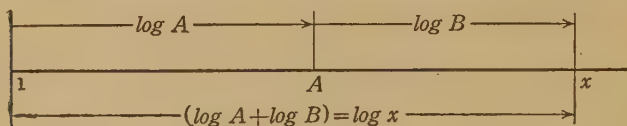


FIG. 1

Suppose we wish to multiply A by B and let $x = AB$. Then by formula (15) $\log x = \log A + \log B$. We look up $\log A$ in a table of logarithms and lay it off to any desired scale of drawing on a straight line. Let this be the length from 1 to A in Fig. 1. To the same scale we lay off the length A to x for $\log B$. Then the sum of these two lengths, 1 to x on the line, is $(\log A + \log B)$ and by the formula this equals $\log x$. So we measure the total length 1 to x on our line and determine the number x to which this length corresponds, by using the drawing scale. This number is the product of A and B and we have thus found $x = AB$ mechanically and without any multiplying.

The logarithmic line begins with 1 instead of 0 (zero) because its length represents the logarithms of numbers while the figures we have marked on the line are the numbers themselves which we are multiplying, and by formula (23) the zero length represents the logarithm of 1.

Using the same sort of scale we find $y = A \div B$ as indicated in Fig. 2. By formula (16) we know that when $y = A \div B$, then $\log y = \log A - \log B$. Using our scale line as before we lay off from 1 the length cor-

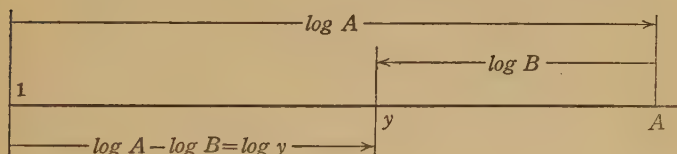


FIG. 2

responding to $\log A$, bringing us to the point A on the scale. From this length we subtract the length $\log B$ by starting at A and laying off $\log B$ in the reverse direction, bringing us to the point y . Since the length 1 to y is now equal to $(\log A - \log B)$ it represents to our chosen scale the logarithm of y . From the drawing scale we then determine the number corresponding to this length and we have the quotient of $A \div B$.

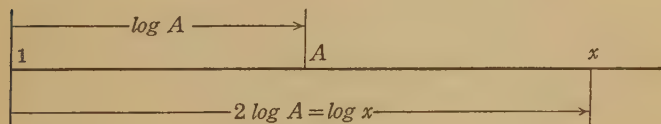


FIG. 3

Suppose now that we wish to find the square of some number A on our scale. By formula (17) we know that $\log(A^2)$ is 2 times $\log A$, so we need only lay off on our

line twice the scale length corresponding to A and see what number it reaches on the line. This is shown in Fig. 3. Here the number $x = A^2$ is found from the length of $\log x$ (1 to x on the line) as explained above. In the same way we could find A^3 by putting $x = A^3$ and finding the length corresponding to $\log x = 3 \log A$. Applying formula (18) to our scale line we can easily find $y = \sqrt{A}$ by laying off as in Fig. 4 the length $\log A$

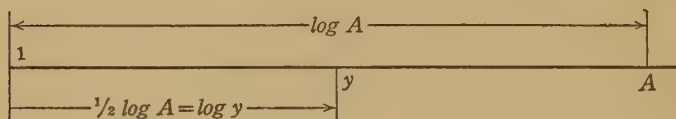


FIG. 4

and then finding the scale number corresponding to $\frac{1}{2} \log A$. By the same method we could just as easily find $y = \sqrt[3]{A}$.

A more rapid method of finding squares and cubes and roots is based on the following considerations: When drawn to the same scale, a line of a certain length which is twice a shorter length represents the logarithm of the square of the number whose logarithm is represented by the shorter length. Therefore when we use two such drawing scales on the same line, or on two lines side by side, a length on the line drawn to the smaller scale will represent the square of a number represented by the same length on the large scale line. On the other hand a length on

the large scale line will represent the square root of the number represented by the same length on the small scale line.

Thus let the two lines of Fig. 4 be drawn separately

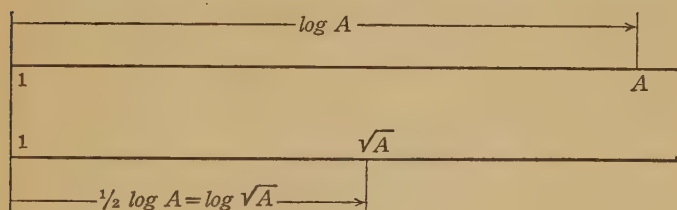


FIG. 5

from the same starting point, as in Fig. 5. Now let the scale of the upper line be reduced so that the length representing $\log A$ on it equals the length representing $\log \sqrt{A}$ on the other scale. This gives us Fig. 6. By

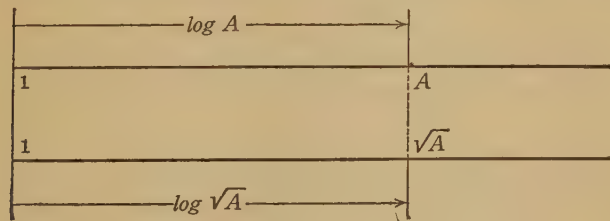


FIG. 6

drawing two such scale lines therefore, and laying a straight edge across them at right angles at a point on the upper line corresponding to any chosen number it will cross the lower line at a point corresponding to the square

root of the number. By choosing a number on the lower scale line its square is similarly found on the upper line.

The processes described in connection with Figs. 1, 2, 6 embody the principles of the slide rule and have here been described and explained at length because of their extreme importance.

11. Application of the Slide Rule Principle. — The details of the operations described and explained in the preceding article are somewhat laborious and their performance is much facilitated if instead of drawing one

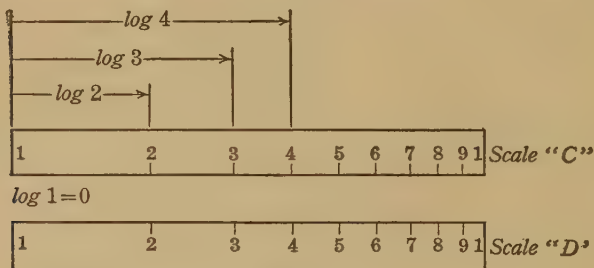


FIG. 7

scale line as in Figs. 1 and 2 and separately measuring off the proper lengths on it, we draw two such lines to the same scale on two strips of wood, laying off to a convenient scale the *lengths* corresponding to the *logarithms* of the numbers, and marking on the lines the *FIGURES* representing the *NUMBERS* themselves. Such an arrangement is represented in Fig. 7.

The scale divisions are not uniform because when the logarithms are in arithmetical progression the numbers are in geometrical progression as in the tables below; and when the numbers are in arithmetical progression the logarithms are in geometrical progression as on the scales in Fig. 7.

Number	=	1,	10,	100,	1000,	10000,	100000,	1000000,
Logarithm	=	0,	1,	2,	3,	4,	5,	6,
Base		10	10	10	10	10	10	10.

Number	=	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,
Logarithm	=	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,
Base		2	2	2	2	2	2	2	2	2	2.

The use of the scales of Fig. 7 in multiplication is shown in Fig. 8 where 2 is multiplied by 3. Thus $2 \times 3 = 6$ is

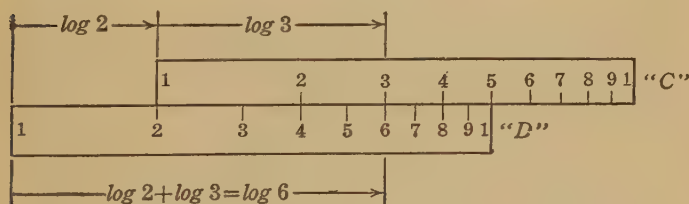


FIG. 8

found by simply placing the beginning of the "C" scale at the multiplicand 2 on the "D" scale and at the multiplier 3 on the "C" scale reading the product 6 on the "D" scale.

The operation described in connection with Fig. 2 is now carried out very easily by placing the two wooden

scales together as in Fig. 9, which shows the division of 8 by 4. Here the divisor 4 on the "C" scale is placed at the dividend 8 on "D" and the quotient 2 is read on "D" at the 1 of "C."

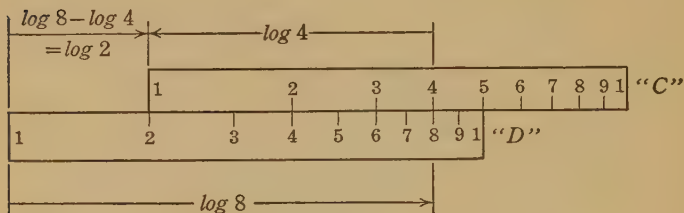


FIG. 9

As the same logarithm table serves for all numbers by shifting the decimal points or changing the characteristics, so in both Figs. 8 and 9 we could, by shifting the decimal points, read 2, 3, 6; 2, 30, 60; 20, 30, 600; .2, .3, .06; .2, .03, .006; or 8, 4, 2; 80, 40, 2; 0.08, .4, .2, etc., as might be necessary. This procedure will be explained in detail later. Also by making the scales of any desired length and subdividing the main divisions decimally, any numbers, whole, fractional or mixed, may be read as closely as desired.

The application of Fig. 6 to the wooden scale or rule is shown in Fig. 10. Here the "D" scale is the same as in Figs. 7, 8, 9 but "A," like the upper line in Fig. 6, is laid off to a scale half as great. The straight edge laid across

the two scales at 9 on "A" crosses "D" at $3 = \sqrt{9}$. Conversely, in line with 6 on "D" is found $6^2 = 36$ on "A."

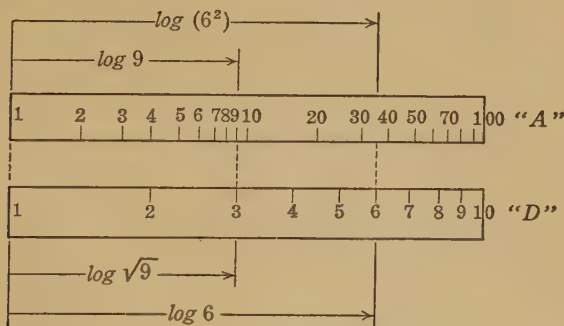


FIG. 10

The scales "A," "C" and "D" are incorporated in the Mannheim Slide Rule as described in the next article.

12. The Standard Mannheim Slide Rule.—The American standard slide rule is that designed by Lieutenant Mannheim and consists of a strip of wood about $10\frac{5}{8} \times 1\frac{1}{8} \times \frac{3}{8}$ inches ($27 \times 2.9 \times 1$ cm.) with a groove cut in one side about $\frac{9}{16} \times \frac{3}{16}$ inches (about 1.4×0.5 cm.) in which slides, with outer faces flush, a second strip of wood about $10\frac{5}{8} \times \frac{9}{16} \times \frac{1}{8}$ inches ($27 \times 1.4 \times 0.4$ cm.), guided by tongues fitting in smaller grooves in the edges of the larger groove. This arrangement is shown in section (not to scale) in Fig. 11 (b) and in plan (not to scale),

with the sliding strip partly drawn out to the right, in Fig. 11 (c). The grooved part is called the "stock" or "rule" and the sliding strip the "slide." On the upper face of each are two divided scales $9\frac{2}{3}\frac{7}{2}$ inches (25 cm.) long, arranged as shown in Fig. 12 and marked on celluloid or ivory facings. These are the A, C and D scales already described and a fourth scale called the "B" scale.

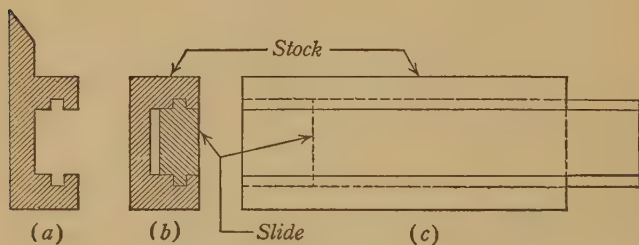


FIG. 11

1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	1	A
1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	1	B
1			2				3		4		5		6		7	8	9	1	C
1			2				3		4		5		6		7	8	9	1	D

FIG. 12

The A and B scales are exactly alike and the same as the A scale in Fig. 10. The C and D scales are exactly alike and the same as in Figs. 8, 9. On the opposite side of the slide are three scales marked "S," "T" and "L."

These scales will be described later. A glass plate with a hair line on its under side which reaches across all the scales at right angles, is attached to the rule and arranged to slide along its full length. This is called the "runner" or "indicator." In most cases the stock bears an extension or "lip" on the side bearing the A scale as shown in section in Fig. 11 (a). This lip carries on its beveled edge an ordinary 10 inch or 25 centimeter rule marked on celluloid or ivory. A complete rule as manufactured by Keuffel & Esser is shown in detail and to scale in Fig. 13 with the B and C scales face upward on the slide.

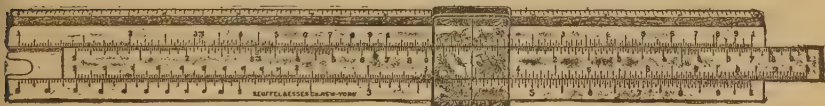


FIG. 13. MANNHEIM SLIDE RULE

Rules with 5, 8, 16, and 20 inch scales are frequently used but the 10 inch (25 cm.) length is the one most used.

B. OPERATION OF THE RULE.

13. Graduation and Reading of the Scales.—As shown plainly in Fig. 13 the C and D scales are alike and divided into 10 main parts, reading 1, 2, 3, 4, 5, 6, 7, 8, 9, 1. These are generally referred to as 1 to 10 although, since the graduation is decimal, they may be taken as .1 to 1, 1 to 10, 10 to 100, etc. The figure 1 at

the left is called the "left index" and that at the right the "right index."

Each of these main divisions is divided into 10 parts, and each of these is in turn subdivided, though not all alike. Thus between 1 and 2 the first subdivisions are divided into 10 parts, between 2 and 4 into 5 parts, and between 4 and 10 into 2 parts. Between 1 and 2 the first subdivisions are themselves numbered 1 to 9 but the others are not usually numbered.

When the main figures are taken as units the first subdivisions are tenths, and between 1 and 2 the second subdivisions are hundredths, between 2 and 4 they are two hundredths, and between 4 and 10, five hundredths. By means of the hair line marker on the runner these smallest subdivisions may themselves be divided and the value of the divisions estimated by the eye. In this way between 1 and 2 ten-thousandths may be estimated directly (the fourth figure); between 2 and 4 hundredths may be estimated directly and with practice five thousandths (the third and sometimes fourth figure); between 4 and 10 hundredths may be estimated directly and in some cases, as between the tenth mark and the five hundredths mark, we may read .025 or .075 giving the third and in special cases the fourth figure. In general we say that on a 10 inch rule we read the fourth figure at the left and the third figure toward the right.

As an example in scale reading consider the setting of the rule shown in Fig. 13. The left index of the C scale is midway between 119 and 120 on D; this is read 1195, "one one nine five," without reference to units or decimal point. Depending on the values and magnitude of the numbers we may be using in a particular case, this may be 1.195, 11.95, 119.5, 1195, 11950, etc., or .1195, .01195, .001195, etc.

Again, the hair line of the runner or indicator crosses the C scale at 346, "three four six." It crosses the D scale between 41 and 415 at a little more than three-fifths of the distance from 41, that is, to three figures, 413, "four one three." Similarly the right index of the D scale is just at the right of 835, that is, at 836, "eight three six."

Comparing the A and D or the B and C scales it is seen that A consists of two duplicate scales each of which runs 1 to 10, or considered as one scale it has three indexes and runs 1 to 10 to 100. Each numbered division is divided into 10 parts and from 1 to 5 these are subdivided, between 1 and 2 into 5 parts and between 2 and 5 into 2 parts. Thus between 1 and 2 the smallest subdivisions are two hundredths, between 2 and 5 they are five hundredths, and between 5 and 10, tenths. Thus with the indicator we can estimate hundredths, though not so accurately from about 8 to 10. With reference

to the values of these numbers and the position of the decimal point the same thing applies to the A and B scales as to C and D. Thus if from the left index of A to the middle index the figures are taken as units, then from the middle index to the right the figures indicate tens and we read 10 to 100. In the same way as for C and D we may also read A and B as 10 to 100 to 1000, etc., or .001 to .01 to .1, .01 to .1 to 1, .1 to 1 to 10, etc. The position of the decimal point on the A and B scales and the relation of these scales to C and D will be discussed in detail farther on in this book and we shall then see that when the units of C or D are chosen then for certain purposes the units of B or A are fixed automatically.

The left hand or first half of the A scale is called A1 and the right hand or second half is called A2. Similarly the left and right halves of B are called B1 and B2. In what follows we shall use these designations.

As examples in reading the A and B scales consider again the "settings" (positions of slide and runner) of the rule in Fig. 13. The left index of B is midway between 142 and 144, at 143, on A. The middle index of A is at 700 on B. The middle index of B is at 143 on A, and the right A index is at 700 on B. The indicator is at 171 on A and at 120 on B.

Just to the right of 3 on A1 and B1 and to the left of 8 on A2 and B2 are two special marks called "gauge points."

These are respectively $\pi = 3.1416$ and $0.7854 = \frac{1}{4} \pi$. Their uses will be explained later.

The descriptions of the scales and reading are given here in detail and they should be carefully studied and practice settings

made and read until they are thoroughly mastered. When

this is done statements of complicated settings and operations are easily understood

and followed without the necessity of detailed descriptions or elaborate diagrams. Thus the setting of the slide and runner may be stated as

<i>A</i>	<i>To 143</i>	<i>Read 171</i>
<i>B</i>	<i>Set 1</i>	<i>R to 120</i>

(a)

<i>C</i>	<i>Set 1</i>	<i>R to 346</i>
<i>D</i>	<i>To 1195</i>	<i>Read 413</i>

(b)

<i>A</i>		<i>Read 171</i>
<i>B</i>		
<i>C</i>	<i>Set 1</i>	<i>R to 346</i>
<i>D</i>	<i>To 1195</i>	

(c)

FIG. 14

follows: "Left C index to 1195 on D, Runner to 346 on C, Read 413 on D"; or "Left B index to 143 on A, Runner to 120 on B, Read 171 on A." These may be diagrammed separately as in Fig. 14 (a) and (b), or in one combination as in Fig. 14 (c).

Forms of statements of settings similar to those given in quotation marks above, or setting diagrams such as those of Fig. 14, will be much used in what follows.

14. Multiplication with the Slide Rule. — The C and D scales of the Mannheim Slide Rule are used in multiplication exactly as indicated with the two detached scales C, D in Fig. 8 but with the actual rule the slide is held in place in the groove of the stock by friction and it is not necessary to press the scales together by hand; also the indicator is a great aid in making accurate settings and readings.

Suppose we wish to multiply 2 by 3. We say “To 2 on D set left 1 of C; Runner to 3 on C; At indicator read 6 on D.” It is to be noticed that the numbers are taken in the order in which they appear in the multiplication as written:

$$2 \times 3 = 6$$

(1) (2) (3)

Except in special cases multiplication is always performed on the C, D scales and the operation is always carried out as follows

- (1) To *Multiplicand* on D set C index.
- (2) To *Multiplier* on C set runner indicator.
- (3) At indicator on D read *Product*.

The operation $2 \times 3 = 6$ is shown in detail in Fig. 8 and

is diagrammed in Fig. 15 (a). Fig. 15 (b) shows multiplication in general.

<i>C</i>	<i>Set 1</i>	<i>R to 3</i>
<i>D</i>	<i>To 2</i>	<i>Read 6</i>

(a)

<i>C</i>	<i>Set 1</i>	<i>R to Multiplier</i>
<i>D</i>	<i>To Multiplicand</i>	<i>Read Product</i>

(b)

FIG. 15

In the case of $2 \times 3 = 6$ the readings are exact and simple. Suppose, however, we wish to multiply 1.195 by 3.46 and read the product correct to the nearest three figures. The operation is:

- (1) To 1.195 on D set C index.
- (2) To 3.46 on C set runner.
- (3) At indicator on D read 4.13.

To simply set one index of C at 1.195 on D is tedious and also liable to error since the black line at 1 has some width. But if we first set the very fine hair line of the indicator at 1.195 on D and then bring up the slide so that the 1 line is exactly under the indicator hair line the setting is generally much more easily and accurately made. Similarly if we have to locate the point 3.46 on C

with the unaided eye and then determine the D scale reading which is in line with this, as is done in Fig. 8 with the 3 and 6, there is not only difficulty in reading but liability to error. It is thus seen that the indicator is at the same time a labor and time saver and also when properly used an insurance against many errors. We shall see later that the indicator is also a necessity in certain forms of settings.

The multiplication $1.195 \times 3.46 = 4.13$ is already set and described in Fig. 13 and diagrammed on C, D in Fig. 14 (b). There we read only the figures 1195, 346, 413. The values of the numbers are determined by the position of the decimal point and when it is once placed in the multiplicand and multiplier its position in the product is determined by those. Thus we know that 1.195×3.46 cannot be 0.413, neither can it be 41.3. A number slightly greater than 1 multiplied by one nearly equal to $3\frac{1}{2}$ must give a product closely equal to 4. So the product can only be 4.13. Again, if our multiplication were 11.95×34.6 it would be nearly 12×35 and the product must be in the neighborhood of 400, that is, 413. So in any case a knowledge of the values of the factors gives at once the approximate value of the product and when the scale reading gives the figures the proper position of the decimal point is known at once. We shall later give definite rules for locating decimal points, after we have

described the operation of the rule, but these should not be used blindly. The results should always be checked by a knowledge of simple arithmetic. As a matter of fact physicists and engineers almost always locate decimal points in slide rule results by inspection.

Another question now arises. If we carry out the multiplication 1.195×3.46 in the ordinary way we get 4.13470. To the nearest three figures this would be written 4.13; thus the slide rule automatically makes our approximations for us. If the objection is made that the result is not exactly correct it must of course be agreed that it is not, but decimal fractions rarely are exact and the conditions of the problem in hand or simply choice will usually determine just how far to carry the approximation to exactness. In the present case the error is $413470 - 413000 = 470$ parts in 413470, that is $\frac{470}{413470} = .001137 = .1137\%$, about $\frac{1}{9}$ of 1% or 1 part in 900. In such operations as those involving counting where figure accuracy is required the slide rule can only be used as an approximate check, but in measurement and in general engineering work where proportional or percentage accuracy is considered, the slide rule is in general sufficiently accurate, reading results to about one tenth of one per cent.

We now consider a slightly different type of multipli-

cation operation. Let us multiply 43.3 by 5.25. This is stated:

- (1) To 43.3 on D set C index.
- (2) To 5.25 on C set runner.
- (3) At indicator on D read *Product*.

We readily set the indicator line at 433 on D and as usual bring up the left C index line to the indicator line and step (1) is complete. When we attempt to bring the

C	Set 1	R to 525
D	To 433	Read 227

(a)

A	To 433	Read 227
B	Set 1	R to 525

(b)

FIG. 16

runner (indicator) to 525 on C, however, we find this to be impossible; that point is beyond the end of D. As we ordinarily say, it "runs off the scale."

We get around this difficulty by the following considerations: Since the logarithmic scale is decimal we may read it either way, that is, call the left index 1 and the right 10 or the right index 1 and the left .1. Thus since we may take either index as 1 we may use either index to set on the multiplicand in multiplication. In the present case then let us try the right C index at 433 on D; we now easily bring the runner to 525 on C. Step (3) of the operation then gives for the product the figures 227. Now $43\frac{1}{3} \times 5\frac{1}{4}$ cannot be 22.7 or 2270; the result is 227. The diagram

of this operation is made in the usual manner in Fig. 16 (a) with "C index" here indicating the right hand index.

Multiplication may be carried out with the A and B scales in precisely the same manner as with C, D, the B scale (on the slide) being used instead of C and the A scale (on the stock) instead of D. Either half of A or B may be used. Thus to apply to either set of scales the statement corresponding to Fig. 15 (b) might read:

- (1) To *Multiplicand* on STOCK set SLIDE index.
- (2) To *Multiplier* on SLIDE set runner indicator.
- (3) At indicator on STOCK read *Product*.

The last multiplication, $43.3 \times 5.25 = 227$, would thus be carried out on A, B as indicated in Fig. 16 (b). The A, B

scales are used for multiplication in certain combination settings in connection with C, D but for straight multiplication, C, D should always be used.

C	Set 1	R to 47.6
D	To 73.5	Read 3500

(a)

C	Set 1	R to 6.78
D	To .1473	Read 1

(b)

FIG. 17

Two sample multiplications are given in Fig. 17 and these, together with those already discussed, and many others, should be practiced until the operations are carried out without hesitation.

$$(a) 73.5 \times 47.6 = 3500 \quad (b) .1473 \times 6.78 = 1.000$$

In order to carry out such a multiplication as $1.5 \times 3 \times 2 = 9$ we may first multiply $1.5 \times 3 = 4.5$ in the usual manner as in Fig. 18 (a) and then multiply $4.5 \times 2 = 9$ as in Fig. 18 (b). It will be noted that when we have read the 4.5 in (a) we ordinarily leave the runner unmoved on 4.5, for 4.5 is now the multiplicand of operation (b) and we ordinarily mark the multiplicand with the runner so as to bring up the slide to set the index accurately. Since we are not to move the runner from the 4.5 then until the slide is reset, we do not need to read the first product 4.5 but may combine the double operation into one double setting and read only the final result. This operation is shown in Fig. 18 (c).

<i>C</i>	<i>Set 1</i>	<i>R to 3</i>
<i>D</i>	<i>To 1.5</i>	<i>Read 4.5</i>

(a)

<i>C</i>	<i>Set 1</i>	<i>R to 2</i>
<i>D</i>	<i>To 4.5</i>	<i>Read 9</i>

(b)

<i>C</i>	<i>Set 1</i>	<i>R to 3</i>	<i>1 to R</i>	<i>R to 2</i>
<i>D</i>	<i>To 1.5</i>			<i>Read 9</i>

(c)

FIG. 18

The double setting is stated as follows:

- (1) To 1.5 on D set C index.
- (2) To 3 on C set runner.
- (3) To runner set C index.
- (4) To 2 on C set runner.
- (5) At indicator on D read 9.

Similarly let us multiply $1.72 \times 2.94 \times 1.265$:

- (1) To 1.72 on D set C index.
- (2) To 2.94 on C set runner.
- (3) To runner set C index.
- (4) To 1.265 on C set runner.
- (5) At indicator on D read 6.40.

Here we did not read or need the intermediate product 5.06 of the first two factors.

In the same way any number of factors may be multiplied as follows. Let

$$a \times b \times c \times d = X$$

- Then
- (1) To a on D set C index.
 - (2) To b on C set runner.
 - (3) To runner set C index.
 - (4) To c on C set runner.
 - (5) To runner set C index.
 - (6) To d on C set runner.
 - (7) At indicator on D read X .

It is to be noted that we read and set the multipliers on the slide only and alternately move the slide and runner, reading only the first factor and the final product on D.

Thus we may state the above operation concisely as follows:

Set slide index to a on D, runner to b on slide, slide to runner, runner to c , slide to runner, runner to d ,
Read X on D;

and this series of operations is repeated for any number of factors $a \times b \times c \times d \times e \times \dots = X$, alternately setting the runner to the factors (after the first) and the slide index to the runner.

15. Division and Reciprocals. — Division is the inverse of multiplication and the slide rule operation for division

<i>C</i>	<i>Set 3</i>	<i>Under 1</i>
<i>D</i>	<i>To 6</i>	<i>Read 2</i>

(a)

<i>C</i>	<i>Set 4</i>	<i>R to 1</i>
<i>D</i>	<i>To 8</i>	<i>Read 2</i>

(b)

FIG. 19

is exactly that described in connection with Fig. 9 or the inverse of that shown in Fig. 8. Thus we can consider that Fig. 8 shows either the operation $2 \times 3 = 6$

or $6 \div 3 = 2$ and by comparison of Figs. 8, 9 with the setting diagram Fig. 15 (a) we see that the diagrams for $6 \div 3 = 2$ and $8 \div 4 = 2$ are those of Fig. 19 above. The operation $8 \div 4 = 2$ may be stated as follows:

- (1) To 8 on D set 4 on C.
- (2) To C index set runner.
- (3) At indicator on D read 2.

If the A, B scales are used it is stated:

- (1) To 8 on A set 4 on B.
- (2) To B index set runner.
- (3) At indicator on A read 2.

In general for either A, B or C, D:

- (1) To *Dividend* on stock set *Divisor* on slide.
- (2) To slide index set runner.
- (3) At indicator on stock read *Quotient*.

It is to be noted that the operation is carried out in exactly the reverse of the order of multiplication, thus:

Multiplicand $\times$ Multiplier = Product

$$\begin{array}{ccccc} 2 & \times & 3 & = & 6 \\ (1) & & (2) & & (3) \end{array}$$

but Dividend $\div$ Divisor = Quotient

$$\begin{array}{ccccc} 6 & \div & 3 & = & 2 \\ (1) & & (2) & & (3) \end{array}$$

It is now at once obvious that the setting of the rule in Fig. 13 may be viewed as that of $413 \div 346 = 1195$ (reading the figures only and ignoring decimal point), which would be stated and diagrammed as:

- (1) To 413 on D set 346 on C.
- (2) To C index set runner.
- (3) At indicator on D read 1195.

C	Set 346	R to 1
D	To 413	Read 1195

FIG. 20

Of course Fig. 13 shows only the first step of the operation, setting 346 on C to 413 on D (by means of the runner). It is seen, however, that the C index is already at 1195 on D and when the runner is brought to 1 on C it will exactly indicate the figures 1195 on D.

The decimal point may now be determined by inspection. Thus if the numbers are 413 and 346 the quotient is obviously slightly greater than 1 and the decimal point is placed to give 1.195. If we are dividing 41.3 by 3.46 it is obvious that the quotient is very nearly $41 \div 3\frac{1}{2}$ or about 12, and when the decimal point is placed we have 11.95.

If Fig. 13 is carefully examined or if the setting of the rule as shown in Fig. 13 is reproduced on the actual slide rule it will be seen that the reading which we have called 413 is slightly greater than 413 but hardly 414; it should be read 4135. If the division $413.5 \div 346$ is carried out by long division the quotient does not come out exact, but to six figures it is 1.19364 and the slide rule error is $119500 - 119364 = 136$ parts in 119364, that is, $\frac{136}{119364} = .001138 = .1138\%$, or about $\frac{1}{9}$ of one per cent.

As a final example let us divide 164 by 34. The statement is

- (1) To 164 on D set 34 on C.
- (2) To C index set runner.
- (3) At indicator on D read *Quotient*.

Step (1) is easily and exactly made; when we attempt step (2), however, only the right C index can be reached. We try this index and the quotient read on D is 4.825, which is exactly correct. So in division there is only one choice of index and we do not have to reset the slide

to determine it. The setting for this division is Fig. 21. Incidentally, this setting furnishes an example of a four-figure scale reading toward the right end of the scale which is easily made. It also gives an example of a slide rule result which is exactly correct. Instead of having an error of about one tenth of one per cent the error is zero.

C	Set 34	R to 1
D	To 164	Read 4.825

FIG. 21

A *reciprocal* is a quotient with the dividend equal to 1. Thus the *reciprocal of a number* is a common fraction having the number as denominator and 1 as numerator and the slide rule furnishes a quick and direct reduction of such fractions to decimal fractions. Thus $\frac{1}{4} = 1 \div 4 = .25$ and this operation on the slide rule is stated in the usual manner as

- (1) To 1 on D set 4 on C.
- (2) To 1 on C set runner.
- (3) At indicator on D read .25.

The diagram is Fig. 22 (a). From this we obviously have the following rule for reducing reciprocals to decimals, or for finding the reciprocal of any number which may itself be a whole number, decimal number or mixed number: "*To D index set the number on C and under C index read the reciprocal.*" The diagram is Fig. 22 (b).

The inverse operation, that of expressing any decimal fraction as a common fraction with the numerator 1, is sometimes useful. From the preceding rule and Fig. 22

<i>C</i>	<i>Set 4</i>	<i>R to 1</i>
<i>D</i>	<i>To 1</i>	<i>Read .25</i>

(a)

<i>C</i>	<i>Set No.</i>	<i>Under 1</i>
<i>D</i>	<i>To 1</i>	<i>Read Recip.</i>

(b)

FIG. 22

we have as the inverse rule: "*To the decimal on D set the C index and the number on C nearest the D index is the denominator.*" Thus for .0237 we find nearly $\frac{1}{42}$.

As a matter of fact, any common fraction may be considered as a division. Thus $\frac{3}{4} = 3 \div 4 = .75$ and we have at once the rule for reducing common fractions to decimals: "*To numerator on D set denominator on C and under C index find decimal.*" This rule is diagrammed in Fig. 23 (a).

The inverse of this rule is also sometimes useful, to express a decimal as a common fraction. Reversing the last rule we have: "*To the decimal on D set the C index and find on C and D the two simple numbers nearest to-*

gether; that on *C* is the numerator and that on *D* the denominator." This rule is diagrammed in Fig. 23 (b). A simple illustration is $.428 = \frac{3}{7}$, very nearly.

<i>C</i>	<i>Set Denominator</i>	<i>Under 1</i>
<i>D</i>	<i>To Numerator</i>	<i>Read Decimal</i>

(a)

<i>C</i>	<i>Set 1</i>	<i>Find Numerator</i>
<i>D</i>	<i>To Decimal</i>	<i>Over Denominator</i>

(b)

FIG. 23

16. The Decimal Point in Slide Rule Multiplication and Division. — We will now give the rules for locating decimal points which have been referred to in several places. This can best be done in connection with illustrative examples.

Thus consider the multiplication $24.3 \times 37 = 900$. We set the left *C* index to 24.3 on *D* and the slide projects from the rule toward the right. The same is the case with $1.88 \times 4.2 = 7.9$ and with $152.5 \times 58.2 = 8875$. Let us write these down together for comparison:

- (1) $1.88 \times 4.2 = 7.9$
- (2) $24.3 \times 37. = 900.$
- (3) $152.5 \times 58.2 = 8875.$

In (1) the sum of the numbers of figures to the left of the decimal points in the factors is 2 and there is one figure to the left of the decimal point in the product. In (2) this sum is 4 and the product has three figures before the decimal point, and in (3) these figures are 5 and 4. In each case the number of figures before the decimal point in the product is one less than the sum of those in the factors. This will be found to be the case whenever the slide projects to the *right* in multiplication.

Consider the product $3.72 \times .242 = .9$; the same rule holds. In the product $.275 \times .32 = .088$ the corresponding sum for the factors is zero and not only are there no digits before the decimal point in the product, but also in one place after the decimal point there is no digit, the place being filled by a cipher (0). In this case the result of the examples (1), (2), (3) above would give $0 - 1 = -1$. Similarly when applied to the multiplication $.041 \times .222 = .0091$ we would have to write the rule as $-1 - 1 = -2$ and there are two places after the decimal point in the product filled by ciphers.

In order to state a rule for the decimal point in concise terms we define: The *characteristic* of a number is the number of digits before the decimal point; the characteristic of a decimal fraction is the number of ciphers immediately after the decimal point and is negative. (It is to be noted that the characteristic of a *number*

is not the same as the characteristic of the *logarithm* of a number.) For example, the characteristic of 1.88 and of 4.2 is 1; the characteristic of 24.3 and of 37. is 2; that of 152.5 is 3 and that of 8875. is 4. The characteristic of .9 is zero; that of .088 is -1 and that of .0091 is -2 , etc. Using this definition we can now easily and concisely express the result of the several multiplications discussed above as a rule in the following form:

- (i) RULE: *When the slide projects to the right in multiplication the characteristic of the product is 1 less than the sum of the characteristics of the factors.*

This rule holds for any number of factors by taking them two at a time; the detailed application to several factors will be given after the rules have been given for two factors and for division.

Consider now the operations:

$$\begin{array}{ll} (1) 73.5 \times 3.13 = 230. & (3) .675 \times .163 = .110 \\ (2) 56.7 \times .45 = 25.5 & (4) 8.33 \times .0021 = .0175 \end{array}$$

In each of these the slide projects to the *left* and examination shows that the characteristic of the product is equal to the sum of the characteristics of the two factors. This gives the rule:

- (ii) RULE: *When the slide projects to the left in multiplication the characteristic of the product equals the sum of the characteristics of the factors.*

Since the operation of division is the inverse of that of multiplication, the dividend taking the place of the

product and the quotient and divisor those of the factors, the two rules (i) and (ii) when reversed give the rules for division. This will be made clearer by the following formulation: Let

M	=	characteristic of Multiplicand
m	=	“ “ Multiplier
P	=	“ “ Product
D	=	“ “ Dividend
d	=	“ “ Divisor
Q	=	“ “ Quotient,

then Rule (ii) gives for

	Multiplication	$P = M + m$
	Division	$D = Q + d$
therefore		$Q = D - d$

and we have the following for division:

- (iv) RULE: *When the slide projects to the left in division the characteristic of the quotient equals the characteristic of the dividend minus that of the divisor.*

Similarly Rule (i) gives for multiplication and division

	$P = (M + m) - 1$
	$D = (Q + d) - 1$
therefore	$Q = (D - d) + 1$

which gives the following:

- (iii) RULE: *When the slide projects to the right in division the characteristic of the quotient equals the characteristic of the dividend minus that of the divisor, plus 1.*

The rules (i), (ii), (iii) and (iv) are combined into a simple chart below which is convenient for reference and easy to remember.

Characteristic of result	Slide LEFT	Slide RIGHT
Multiplication	Sum of Characteristics of 2 Factors	Sum - 1
Division	Characteristic of Dividend - that of Divisor	Difference + 1

In order to see how these rules are applied in the multiplication of three or more factors consider the operation $4.2 \times 2.3 \times 19.3 = 186.5$. This may be considered as carried out in two steps as follows: $(4.2 \times 2.3) \times 19.3 = 186.5$. Referring to the chart above, since in the first setting the slide projects to the right we have as the characteristic of the product of the first two factors (though we do not read the product), $1 + 1 - 1 = 1$ and when this product is multiplied by the 19.3 the slide projects to the left and to the result 1 for the first step we add, according to the chart, the characteristic of 19.3, namely 2, and the final result is 3, which is the characteristic of the complete product 186.5. Again consider the operation $58.5 \times 4.65 \times 7.54 = 2050$. Here the slide projects to the left at each step of the operation and the characteristic of the product is the sum of the characteristics of all three factors. Comparing these two

examples and trying out others of the same or greater number of factors it is seen at once that the decimal point is located in products of several factors in the following manner:

- (v) **RULE:** *To find the characteristic of the product of several factors add the characteristics of all the factors, and subtract 1 for each time that the slide projects to the right in carrying out the successive steps of the complete setting.*

The rules developed above for the location of decimal points in slide rule multiplication and division always give correct results but should not be used blindly or mechanically. With practice it will generally be found that decimal points can be located about as quickly and easily, and with greater appreciation of the significance of the results, by simple inspection. As mentioned once before, physicists, chemists and engineers generally use the method of inspection.

17. Combined Multiplication and Division. — After the operations of multiplication and division are mastered the combination of the two is simple. Thus consider the operation $(3 \div 2) \times 5 = 7.5$. The complete operation here consists of the two separate operations $3 \div 2 = 1.5$ and $1.5 \times 5 = 7.5$ and is diagrammed in two parts in Fig. 24.

Since in (b) we must set the C index to 1.5 on D and then bring the runner to 5 on the slide, and since at the

end of the operation (a) the C index is already at 1.5 on D we need not stop to note the intermediate result 1.5

<i>C</i>	<i>Set 2</i>	<i>R to 1</i>
<i>D</i>	<i>To 3</i>	<i>Read 1.5</i>

(a)

<i>C</i>	<i>Set 1</i>	<i>R to 5</i>
<i>D</i>	<i>To 1.5</i>	<i>Read 7.5</i>

(b)

FIG. 24

but may combine (a) and (b) and obtain the final result directly. This is done in Fig. 25 (a).

<i>C</i>	<i>Set 2</i>	<i>R to 5</i>
<i>D</i>	<i>To 3</i>	<i>Read 7.5</i>

(a)

<i>C</i>	<i>Set 243</i>	<i>R to 38.4</i>
<i>D</i>	<i>To 19.45</i>	<i>Read 3.21</i>

(b)

FIG. 25

As an example let us find the combined result $(19.45 \div 243) \times 38.4 = 3.21$. This is done in Fig. 25 (b). The result is pointed off in accordance with the chart given in the

last article. Thus for the step $19.45 \div 243$ the characteristic is $2 - 3 = -1$ and this used in the second part, $(19.45 \div 243) \times 38.4$ gives $-1 + 2 = 1$ as the characteristic of the final result, 3.21. These steps in the procedure of locating the decimal point may be combined into a single rule after the manner of Rule (v), as may also be done in the cases of multiple factors and divisors in the several cases discussed below, but the multiplication of rules tempts the student to memorize them and this is not a good plan. If rules are to be used, it is better to master the two rules given in the chart in the last article and when these are thoroughly understood to combine them in every case to suit the particular setting in hand.

The examples $(3 \div 2) \times 5$ and $(19.45 \div 243) \times 38.4$ may also be written as $(3 \times 5) \div 2$ or $(19.45 \times 38.4) \div 243$ but when carried out in this order the setting is more complicated and it is always better to perform the division first. For this purpose and for the purpose of extending the method to any number of factors and divisors it is better to write $(3 \div 2) \times 5$ in the form $\frac{3 \times 5}{2} = 7.5$, or in general for any numbers whatever,

$$\frac{ac}{b} = X. \quad (24)$$

This operation is diagrammed in Fig. 26 (a).

<i>C</i>	<i>Set b</i>	<i>R to c</i>
<i>D</i>	<i>To a</i>	<i>Read X</i>

(a)

<i>C</i>	<i>Set b</i>	<i>R to c</i>	<i>d to R</i>	<i>R to e</i>
<i>D</i>	<i>To a</i>			<i>Read X</i>

(b)

FIG. 26

The operation corresponding to formula (24) is stated as:

- (1) To *a* on D set *b* on C.
- (2) To *c* on C set runner.
- (3) At indicator on D read *X*.

Comparing this with the statement of the operations of multiplication and division in Articles 14 and 15 it is seen that the combined operation requires no more steps than each of the separate operations. This advantage is even more apparent in the setting corresponding to the operation

$$\frac{ace}{bd} = X \quad (25)$$

which is diagrammed in Fig. 26 (b) and stated as follows:

"To *a* on D set *b* on C, runner to *c* on slide, slide *d* to runner, runner to *e*, at runner on D read *X*."

By alternately moving the slide for divisors and the runner for multipliers this procedure applies to any number of quantities occurring in expressions like (25).

The decimal point in X is located by application of the rules of Article 16 as in the case of $\frac{19.45 \times 38.4}{243}$ above.

18. Proportion. — If we multiply both sides of the equation (24) by b we get

$$ac = bX \quad (26)$$

and since this says that the product of a and c equals the product of b and X , a and c must be the means and b and X the extremes of a proportion, or vice versa. That is, either

$$b : a :: c : X \quad \text{or} \quad a : b :: X : c \quad (27)$$

Thus the slide rule setting of Fig. 26 (a) will solve any proportion for any member X when the other three members are known. This is explained in the following paragraph.

The proportions (27) may also be written in the form

$$\frac{b}{a} = \frac{c}{X}, \quad \frac{a}{b} = \frac{X}{c}, \quad \text{or}$$

$$\frac{b}{a} = \frac{c}{X}, \quad \frac{a}{b} = \frac{X}{c}. \quad (28)$$

Now these expressions are simply transformations of (26); other transformations are

$$\frac{X}{a} = \frac{c}{b}, \quad \frac{c}{X} = \frac{b}{a}. \quad (29)$$

The setting of Fig. 26 (a) may be expressed simply as $\frac{C}{D} \left| \frac{b}{a} \right| \frac{c}{X}$ where the capitals C, D indicate as usual the slide rule scales on which the numbers a, b, c, X are read, and it is seen at once that the positions of the members of the proportion here are precisely the same as in the first of the two equations (28). In the same way we can set up each of the other three forms of the proportion expressions in (28) and (29).

Thus we have the general rule: "*Write any proportion $a : b :: c : d$ in the form $\frac{a}{b} = \frac{c}{d}$ and set up the two known members which occur together in the same relation on the C, D scales. The unknown is then found in line with the third known member.*" Thus the general form for pro-

C	a	c
D	b	d

FIG. 27

portion is Fig. 27 and the unknown member is found in the same position on the rule that it occupies in the proportion.

19. Squares, Cubes and Roots. — In connection with Figs. 10 and 13 we have already explained the meaning of the A scale and its relation to D. The B scale is a duplicate of A and its relation to C is the same as that of A to D. Thus squares and square roots may be found

on B, C on the slide or A, D on the stock but A, D are generally more convenient and will here be referred to constantly.

In order to find the square of any number directly and by the use of the runner alone without the slide, simply set the runner hair line (indicator) on the number on D and at the indicator on A read its square. In order to find the square root of any number simply set the indicator to the number on A and at the indicator on D read its square root. These operations are so simple that they require no diagram or formal statement. The choice of scale (left or right) on A and the location of decimal points will require explanation but the settings and readings themselves offer no difficulties. Thus in Fig. 10 the two settings may be viewed as there indicated, namely, the finding of $\sqrt{9} = 3$ by reading from A to D and $6^2 = 36$ reading from D to A, or they may be viewed as the operations $3^2 = 9$, D to A, and $\sqrt{36} = 6$, A to D. Similarly in Fig. 13 the setting of the runner shows on A, D $\sqrt{17} = 4.13$ or $(4.13)^2 = 17$, and on B, C $\sqrt{12} = 3.46$ or $(3.46)^2 = 12$. Also, using the left C and B index line as an indicator we have on A, D $\sqrt{143} = 1.195$ or $(1.195)^2 = 1.43$.

Concerning the decimal point and the choice of scale on A we may observe that the three settings already referred to may be read as we have written them above, namely,

$$\sqrt{17} = 4.13 \quad \sqrt{12} = 3.46 \quad \sqrt{1.43} = 1.195$$

$$(4.13)^2 = 17 \quad (3.46)^2 = 12 \quad (1.195)^2 = 1.43$$

or they may equally well have the following meaning:

$$\sqrt{1700} = 41.3 \quad \sqrt{1200} = 34.6 \quad \sqrt{143} = 11.95$$

$$(41.3)^2 = 1700 \quad (34.6)^2 = 1200 \quad (11.95)^2 = 143,$$

or even

$$\sqrt{.17} = .413 \quad \sqrt{.12} = .346 \quad \sqrt{.0143} = .1195$$

$$(.413)^2 = .17 \quad (.346)^2 = .12 \quad (.1195)^2 = .0143.$$

Thus whatever value we assign to the numbers on D, 1 to 10, 10 to 100, 100 to 1000, etc., or .1 to 1, .01 to .1, .001 to .01, etc., the numbers on A are their squares. Conversely, if the A numbers are properly assigned their values the C numbers are their square roots. It must be noted however that if the left index is taken as 1 the right index is 100 and not 10; the middle index is 10. Thus running from left to right the A scale will read 1 to 10 to 100, 100 to 1000 to 10,000, etc. and we cannot take the left A index as 10 for $\sqrt{10} = 3.16$ and 3.16 is not at the left index of D. In squaring numbers on D this question does not arise; we simply set the runner on the number on D and read its square on A. Thus numbers 1 to 3.16 on D have squares 1 to 10 on A, 3.16 to 10 on D have squares 10 to 100 on A. Coming back and starting again at the left index of D numbers 10 to 31.6 have squares 100 to 1000 on the left half of A, 31.6 to 100 on D have squares 1000 to 10,000 on the right half of A,

etc. Thus when we know the position of the decimal point in a number on D the decimal point is automatically located in its square on A.

The same applies to decimal fractions. Reading the A, D scales right to left, if D runs 1 to .1, A runs 1 to .1 to .01; when D runs .1 to .01, A runs .01 to .001 to .0001, etc., and again the decimal point is automatically fixed in squares.

The proper half of A is selected for taking square roots by reversing the above considerations on squares of numbers. Thus for numbers between 1 and 10 use the left side of A and the root is between 1 and 3.16; between 10 and 100, right side of A and the root is between 3.16 and 10, etc. This method of selection does very well for fairly small numbers even though it requires a back-and-forth check or count of the scale indexes. It will of course also apply for very large numbers or very small decimal fractions but in such cases the counting and checking is very tedious. General rules can, however, be framed from these considerations which cover both the choice of A scale and the pointing off of squares and roots.

In extracting the square root of a number in arithmetic we begin at the decimal point and proceeding toward the left mark off the digits in pairs. Thus we write 144 as 1,44 and there is one digit before the decimal point in the root for each such group or part of a group in the

number. As examples, $\sqrt{1,44.} = 12.$ $\sqrt{56, 25.} = 75.$
 $\sqrt{1, 46, 41.} = 121.$ $\sqrt{81.} = 9.$ With decimal fractions
 we do the same, proceeding toward the right. Thus
 $\sqrt{.01, 46, 41} = .121$ but $\sqrt{.14, 64, 10} = .383$ approximately,
 etc. We do the same when using the slide rule. Then
 after marking the groups toward the left in numbers
 greater than 1 we have this

RULE: If the last group contains one figure use A1 for finding the square root, if it contains two use A2. In either case the characteristic of the square root read on D equals the number of groups in the given number.

As examples of the use of this rule consider the example discussed above. Thus $\sqrt{1,44.}$ is set on A1 and the root is 12. Similarly $\sqrt{29.8} = 5.45$ and the 29.8 is set on A2, and $\sqrt{12345.6}$ is set on A1 and the root is 111, approximately. For decimal fractions we obtain the following

RULE: After pairing off the digits to the right of the decimal point, at first disregard the groups immediately following the point which contain only ciphers. If in the first group containing other figures the first figure is a cipher use A1. If the first figure is not a cipher use A2. In the root there is one cipher immediately after the decimal point for each group consisting wholly of ciphers in the given number.

Reversing the above discussion and rules for square roots we have for the decimal point in squares:

RULE: If the square is on A1 the characteristic of the square is 1 less than 2 times that of the number, if on

A2 it is twice that of the number. This applies to both positive and negative characteristics.

A number may be squared more accurately though less conveniently by simply multiplying it by itself on the C, D scales. The decimal point is then located in the square by the use of the rule for multiplication. Similarly,

A	<i>Read a^2</i>	<i>To a^2</i>	<i>Read a^3</i>
B		<i>Set 1</i>	<i>R to a</i>
C			
D	<i>Above a</i>		

(a)

A		<i>Read a^3</i>
B		<i>R to a</i>
C	<i>Set 1</i>	
D	<i>To a</i>	

(b)

FIG. 28

reversing the operation of squaring, the square root may be found by setting the runner on the number on D and moving the slide until the same number is at the indicator on C and at the C index on D; this number is the square root of the original number.

The operation of finding the cube or cube root of a

number involves the B scale as well as A, D. To explain this write

$$a^3 = a^2 \times a.$$

With the number a set on D we can find a^2 on A as explained above. Using the B, A scales in the manner described in Article 14 we can then multiply a^2 by a and so find a^3 . The complete operation is diagrammed in Fig. 28 (a).

It is to be noted, however, that when the B index is set at a^2 on A the C index is at a on D, so the C index may be set at a on D in the first place and the runner set at a on B without taking the intermediate reading a^2 on A. Thus the operation requires only one setting and with a set on D gives a^3 on A directly. The setting is given in Fig. 28 (b) and may be stated as follows: To find a^3 ,

- (1) To a on D set C index,
- (2) To a on B set runner,
- (3) At indicator on A read a^3 .

As an example $(4.1)^3 = 69$ is diagrammed in Fig. 29 (a). A second example is $(8.3)^3 = 571$, Fig. 29 (b).

If the left C index is set at 8.3 on D in the last example just given, it is found that 8.3 cannot be reached with the runner on either half of B. When the right C index is used, however, the runner may be set at 8.3 on either half of B and the reading on A is the same at both places, namely 571. In the first example, $(4.1)^3 = 69$, either

C index may be used. If the left index be used the left half of B is used with the right half of A and vice versa,

A		<i>Read 69.0</i>
B		<i>R to 4.1</i>
C	<i>Set 1</i>	
D	<i>To 4.1</i>	

(a)

A		<i>Read 571.</i>
B		<i>R to 8.3</i>
C	<i>Set 1</i>	
D	<i>To 8.3</i>	

(b)

FIG. 29

but the result is the same in both cases. Thus in different cases different combinations will be necessary but the diagram Fig. 28 (b) and the statement describing that setting will always hold good.

The decimal point is found in cubes of numbers by applying in succession the rules given above in this article for squares and in Article 16 for multiplication, or the following rule may be used:

RULE: *For numbers set on C to the left of 215 the characteristic of the cube is 2 less than three times that of the number; for numbers on C 215 to 465 it is*

1 less than three times that of the number; to the right of 465 on C, three times that of the number.

The finding of cube roots is the reverse operation of that of finding cubes. Thus if we take the setting for a^3 in Fig. 28 (b) as that of finding $\sqrt[3]{a^3} = a$ we notice that a^3 is set on A with the indicator and the slide is in such a

A	<i>R to a</i>	
B	<i>Find $\sqrt[3]{a}$</i>	
C	<i>and</i>	<i>At 1</i>
D		<i>Find $\sqrt[3]{a}$</i>

(a)

A	<i>R to 91</i>	
B	<i>Find 4.5</i>	
C	<i>and</i>	<i>At 1</i>
D		<i>Find 4.5</i>

(b)

FIG. 30

position that we have the same number at the indicator on B and at the C index on D. Thus we can diagram the cube root setting $\sqrt[3]{a}$ as in Fig. 30 (a) and state it as follows:

- (1) Runner to a on A,
- (2) Move slide until same number is at indicator on B and at C index on D,

(3) Runner to C index,

(4) At indicator on D read $\sqrt[3]{a}$.

The decimal point in cube roots is located by the following rule:

RULE: Mark off the number whose cube root is to be found, in groups of three figures from the decimal point. The characteristic of the root equals the number of groups in the original number.

A rule which applies to decimal fractions will be given later in connection with a cube scale.

The A and B scales may also be used in connection with C and D to find the squares and square roots of products and quotients and the products and quotients of squares and roots by other numbers. Consider the operation $(ab)^2 = X$. $(a \times b)$ is found as usual on D and immediately above this is its square on A. Similarly $(a \times b)$ may be found on A and $\sqrt{ab}$ is immediately below on D. Also $(a \div b)^2$ is on A above the quotient $(a \div b)$ which is found on D in the usual manner and $\sqrt{(a \div b)}$ is on D just below $(a \div b)$ on A. In all these it is not necessary to read the intermediate product or quotient but when the runner is set as if to read it on D or A the square or square root is read just above or below, as the case may be.

When the square or square root of any product, quotient or given number is found on A or D as explained above, the product or quotient of this result by any other number

may then also be found at once by simply moving the runner to the multiplier on B or C, or by setting the divisor on B or C to the runner in the position just found. If the operations for multiplication, division, squares and roots already given have been understood and practiced it is not necessary to give here settings or examples of these combined operations.

Since by formula (3) in Article 8, $a^4 = (a^2)^2$, the fourth power can be found by taking the square of the square. By (1) $a^5 = a^3 \times a^2$ and the fifth power is the product of the square and cube. Similarly $a^6 = (a^3)^2$ or $(a^2)^3$; $a^7 = a^4 \times a^3$; $a^8 = (a^4)^2$; $a^9 = (a^3)^3$; $a^{10} = (a^5)^2$; and so on.

Also by (9) $a^{2/3} = \sqrt[3]{a^2}$ and when a^2 is read on A from D then $\sqrt[3]{(a^2)} = a^{2/3}$ is found at once. Similarly $a^{3/2} = (\sqrt{a})^3$ and reading $\sqrt{a}$ on D and cubing the result $a^{3/2}$ is obtained at one setting.

19b. Use of Gauge Points on A and B Scales. — It has been pointed out that the gauge points marked on the A and B scales are at $\pi = 3.1416$ and $\frac{1}{4}\pi = 0.7854$. These are used in calculations involving the circumferences and areas of circles and in any other calculations involving π or $\frac{1}{4}\pi$ as factors or divisors.

We have already seen in Article 14 how A and B are used for multiplication. Thus to find $X = ab$ set B index to a on A and at b on B read X on A. Similarly the

formula for the circumference c of a circle of diameter d is $c = \pi d$ and hence to find c set B index at π on A and at d on B read c on A. Similarly $d = \frac{c}{\pi}$ and to find the diameter when the circumference is known set π on B to c on A and at the B index read d on A. This calculation may also of course be carried out on C and D, and probably more accurately, but less conveniently.

The area of any circle of radius R is $a = \pi R^2$. Therefore to find the area when the radius is known set the C index to radius on D (which squares R on A) and at π on B read area on A. Similarly $a = \frac{1}{4} \pi d^2$ and to find the area when the diameter is known set C index to diameter on D and at the gauge point .7854 on B read the area on A. Both of these operations may of course be reversed to find $R = \sqrt{\frac{a}{\pi}}$ or $d = \sqrt{\frac{a}{.7854}}$ when the area is known.

In the same manner π or $\frac{1}{4} \pi$ may very conveniently be used as factors or divisors in calculations involving these numbers with any other numbers or squares.

20. The Inverted Slide. — If the slide is removed from the rule and replaced with ends reversed but with the same face (B, C scales) showing it is said to be “inverted.” With the slide inverted the B scale is contiguous to D and C to A. The numbers on B, C have their usual significance and the usual decimal point rules

apply. The use of the inverted slide simplifies certain operations and such uses will be briefly described. The same remarks apply to A and B as to D and C.

(i) *Reciprocals*. — Bring all indexes at each end into alignment with slide inverted. Any two numbers on C and D under the hair line of the indicator are now reciprocals, that is, their product equals 1 (or 10, 100, 1000, etc., with shifted decimal points). Thus to reduce any fraction of the form $\frac{1}{a}$, $\frac{10}{a}$, etc., to a decimal set the runner at the number on D (or C) and read the decimal under the indicator on C (or D).

(ii) *Multiplication and Division*. — Since in ordinary multiplication numbers on the C and D scales are in the same position and numbers read on C are the same as those on D, while now with the slide inverted C numbers are the reciprocals of those on D, it follows that if we set the index of the inverted slide on a D number and set the runner to a C number we are multiplying the D number by the reciprocal of the C number, that is, dividing by the C number. Thus the operations of multiplication and division are interchanged when the slide is inverted.

Suppose it is necessary to divide the same number by each of a series of different numbers. Ordinarily this requires a complete setting (slide and runner) for each division. With the inverted slide it is only necessary to

set the slide index to the dividend on D and then set the runner on each of the divisors on C in turn and read the corresponding quotients on D. This follows from the fact that $D \div d = D \times \frac{1}{d}$.

The inverted slide also enables us to find all the pairs of corresponding factors of any given number. Thus the number $144 = 16 \times 9$, 18×8 , 24×6 , 30×4.8 , 36×4 , 48×3 , 60×2.4 , or 72×2 , etc. This is shown in Fig. 31 with the slide inverted.

<i>C Inv.</i>	<i>Set 1</i>	<i>Read 9</i>	8	6	4.8	4	3	2.4	2
<i>D</i>	<i>To 144</i>	<i>R to 16</i>	18	24	30	36	48	60	72

FIG. 31

In order to find the factor pairs for any number, therefore, set the *Inverted C* index to the number on D. The runner at any factor on D then gives the corresponding factor on C.

(iii) *Inverse Proportion*. — Inverse proportion is a proportion in which two of the members are reciprocals of numbers instead of the numbers themselves. Thus if the direct proportion is

$$\frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{b} = \frac{\left(\frac{1}{c}\right)}{\left(\frac{1}{d}\right)} \text{ or } \frac{a}{\left(\frac{1}{c}\right)} = \frac{b}{\left(\frac{1}{d}\right)} \quad (30)$$

is the inverse proportion. It may be briefly described by saying that while "in direct proportion more requires more and less requires less, in inverse proportion more requires less and less requires more." An illustration of a direct proportion is the problem: "If $2\frac{1}{2}$ tons of material costs \$35.50 what will $6\frac{3}{4}$ tons cost?" which is written $\frac{2.5}{35.5} = \frac{6.75}{X}$ and solved on the rule with slide in normal position as in Fig. 32 (a). A problem in inverse pro-

<i>C Inv.</i>	2.5	6.75
<i>D</i>	35.5	$x=95.75$

(a)

<i>C Inv.</i>	6	9
<i>D</i>	4	$x=2.67$

(b)

FIG. 32

portion is: "If 6 men can do a job in 4 days how long will it take 9 men to do it?" This is written in the form

$$\frac{6}{\left(\frac{1}{4}\right)} = \frac{9}{\left(\frac{1}{X}\right)}$$

and solved on the rule with inverted slide as in Fig. 32 (b). By comparing the formulas (30) and the settings Fig. 32 any inverse proportion is readily solved.

(iv) *Cube Roots*. — When the slide is inverted the positions of B and C are interchanged. The setting of Fig. 30 (b) therefore becomes Fig. 33 (a) and the statement of the operation for $\sqrt[3]{a}$ is:

- (1) To a on A set inverted C index,
- (2) Move runner until indicator stands on same number on B and D,
- (3) At indicator on D read $\sqrt[3]{a}$.

In Fig. 33 (b) is shown the setting for $\sqrt[3]{216} = 6$.

A	To a	
C	Set 1	
B		At $\sqrt[3]{a}$
D		Read $\sqrt[3]{a}$

(a)

A	To 216	
C	Set 1	
B		At 6
D		Read 6

(b)

FIG. 33

21. **The Scale of Equal Parts (L) and Its Use.** — The L scale is the scale of equal divisions which runs along the center of the reverse face of the slide. (The slide is said

to be "reversed" when it is removed from the rule and replaced with faces interchanged but with indexes not interchanged.) When the slide is reversed and the indexes aligned the L scale runs from left to right and reads 0 to 1. The numbered divisions being tenths the larger subdivisions are hundredths and the smaller are two thousandths. Thus the L scale can be read directly to thousandths. A decimal point is always to be placed before the readings taken from this scale.

The L scale is shown at the center of the slide in Fig. 34 below. Comparing the L and D scales it is seen that they are related in the same manner as the rows of logarithms and numbers in the tables in Article 11. The numbers on L are therefore the logarithms of the numbers on D when the D indexes are 1 and 10, and the L indexes are in accordance with the formulas (23) in Article 9. When the D indexes are taken as 10 and 100, 100 and 1000, or 0.1 and 1, 0.01 and 0.1, etc., the figures on L are the mantissas of the logarithms of the numbers on D and the characteristics of the logarithms are found in the usual way (one less than the characteristic of the number, when the characteristic of the number is defined as above in Article 14).

Thus with the slide reversed and aligned we set the runner at any number on D and at the indicator read the mantissa of its logarithm on L, and vice versa. The L

and D scales taken together therefore are the equivalent of a three-place table of logarithms. On this scale the base is 10, that is, the logarithms are common logarithms.

The L scale may also be used directly with the slide in the normal position by means of an indicator line at one end of the rule on the reverse side. In order to find logarithms in this manner draw the slide out toward the right until the number on C whose logarithm is desired is at the right D index. The logarithm (mantissa) is then on L at the indicator line on the reverse side of the rule and is read by simply turning the entire rule over. To find anti-logs set the mantissa on L at the indicator on the reverse side and read the anti-log on C at the right D index.

(NOTE. On some slide rules the L scale runs from right to left. To read logarithms or anti-logs on such a rule with reversed slide the slide must also be inverted. To use the slide in its normal position, when it is drawn out toward the right, the left C index is set at the number on D and the logarithm (mantissa) is read on the reverse indicator.)

22. The Sine Scale (S) and Its Use. — Besides the L scale there are also two other scales on the reverse face of the slide, the S and T scales. When the slide is reversed

and the indexes aligned the S scale lies along A and T along D. This arrangement is shown in Fig. 34.



FIG. 34

The lengths laid off on S represent the logarithms of the sines of the angles 34 minutes to 90 degrees to the same plotting or drawing scale as that on which lengths on A represent the logarithms of the numbers 0.01 to 1. Thus with the indexes in alignment angles marked on S have their sines on A and vice versa. From 35' to 10° each of the smallest scale divisions on S represents 5'; from 10° to 20° each represents 10'; from 20° to 40°, 30'; 40° to 80°, 1°; and from 80° to 90°, 10°. Due to the fact that between 80° and 90° the sine changes very slowly as the angle changes the scale is greatly condensed in this range and the figures are not printed on the scale for 80° and 90°.

The characteristic of the natural sines read on A1 directly (angles 34.3' to 5° 45') is - 1; that is, the figures are written with a cipher between the decimal point and the first digit. The characteristic of the sines read directly on A2 is zero. Thus to find $\sin 2^\circ 47'$ set the

runner at $2^{\circ} 47'$ on S and at the indicator on A1 read .0485. Similarly on A2 $\sin 13^{\circ} 25' = .232$.

For reading sines of angles less than $34.3'$ special gauge points are used. Just at the left of 2° on S is one of these marks, indicated by the minute sign ($'$); it is the "minutes" gauge point. Similarly the seconds gauge point ($''$) is just at the right of $1^{\circ} 10'$ line on S. To find the sine of any number of minutes set the minutes gauge point on S to the number of minutes on A and at the S index read the sine on A. To find the sine of any number of seconds set the seconds gauge point on S to the number of seconds on A and at the S index read the sine on A. These gauge points apply to both A1 and A2.

The characteristic of sines of angles $34.3'$ down to $3.43'$ is -2 ; from $3.43'$ to $.343'$ or $20.6''$ it is -3 ; from $20.6''$ to $2.06''$ it is -4 ; and from $2.06''$ to $.206''$ it is -5 . As a convenient aid in locating the decimal point it should be remembered that $\sin 1' = .000292$ (about 3 zeros 3) and that $\sin 1'' = .00000485$ (about 5 zeros 5).

The S scale may be used for multiplication in connection with A in the same manner as B is used. Thus to find $X = a \sin b$:

- (1) To a on A set S index,
- (2) To b on S set runner,
- (3) At indicator on A read X .

This is diagrammed in Fig. 35 (a). In Fig. 35 (b) is set the multiplication $4.75 \sin 8^\circ 40' = .715$.

A	To a	Read x
S	Set 1	R to b

(a)

A	To 4.75	Read .715
S	Set 1	R to $8^\circ 40'$

(b)

FIG. 35

The decimal point is located in accordance with the usual rules, the characteristic of the sine factor being known from the statements given above. Division is also carried out in the usual manner, the dividends being numbers on A and the divisors being sines of angles on S.

The S scale may also be used to find sines and anti-sines with the slide in its normal position, by taking advantage of the indicator on the reverse side of the rule as described in Article 21 in connection with the L scale. Thus if the slide in normal position be drawn out toward the right, any angle at the reverse indicator on S has its sine at the right A index on B, and vice versa.

Since $\cos A = \sin(90^\circ - A)$ the cosine of any angle is found by setting $(90^\circ - \text{angle})$ on S and reading the sine on A by either method described above.

With the sine or cosine known we find the secant or cosecant from the relations

$$\sec = \frac{1}{\cos}, \quad \csc = \frac{1}{\sin}$$

by using the method for reciprocals, explained in Article 15. Thus with the slide in normal position set any angle A at the reverse indicator on S; the cosecant is then at the B index on A. Similarly set $(90^\circ - A)$ and the secant of A is at the B index on A.

The use of the S scale in the trigonometric solution of triangles will be explained in Chapter III

23. The Tangent Scale (T) and Its Use. — The lengths laid off on the T scale represent the logarithms of the tangents of the angles $5^\circ 43'$ to 45° to the same drawing scale as that on which the lengths on D represent the logarithms of the numbers .1 to 1. Thus with slide reversed and indexes aligned (Fig. 34) angles on T have their tangents on D, and vice versa. From $5^\circ 45'$ to 20° each of the smallest scale divisions on T represents $5'$; from 20° to 45° each represents $10'$. The characteristic of the natural tangents read on D directly is zero, that is, the figures read are immediately preceded by a decimal point.

Thus tangents are found by using T and D together in the same manner as S and A are used for sines. Multi-

plication and division of numbers by tangents of angles $5^{\circ} 43'$ to 45° is carried out by using T and D in the same manner, including characteristics, as C and D are used in the multiplication and division of numbers, or as S and A are used with sines.

Tangents of angles smaller than $5^{\circ} 43'$ are not given on T and D. Since to three figures, however, the tangent of an angle in this range is equal to its sine the S and A scales are used for tangents of these angles, and in exactly the same manner as for sines in every respect, including the characteristic.

The T scale may also be used in connection with the reverse indicator to find tangents and anti-tangents with the slide in its normal position. Thus if the slide in its normal position is drawn out toward the right, any angle at the reverse indicator on T has its tangent on C at the right D index, and vice versa.

Tangents of angles 45° to 90° are not given directly on T, D but since for any such angle A

$$\tan A = 1 \div \tan(90^{\circ} - A),$$

that is, the reciprocal of $\tan(90^{\circ} - A)$, it is only necessary to find $\tan(90^{\circ} - A)$ on C with the slide in normal position as described above, and then to find the reciprocal of the number on D as explained in Article 15. Thus set at the indicator on the reverse side of the rule the angle $90^{\circ} - A$. Then since $\tan(90^{\circ} - A)$ is at the right D index on C,

$\tan A = 1 \div \tan(90^\circ - A)$ is at the left C index on D. This setting is therefore stated as follows:

- (1) To reverse indicator set $(90^\circ - A)$ on T,
- (2) To C index set runner,
- (3) At indicator on D read $\tan A$.

It is diagrammed in Fig. 36 (a). Tangents of angles greater than 45° are greater than 1; the decimal point is located in accordance with the method of pointing off reciprocals. A numerical example, $\tan 52^\circ 35' = 1.307$, is diagrammed in Fig. 36 (b).

<i>T</i> (normal)	$(90^\circ - A)$ to Reverse Indicator	
<i>C</i>		<i>R to 1</i>
<i>D</i>		<i>Read tan A</i>

(a)

<i>T</i> (normal)	$37^\circ 25'$ to Reverse Indicator	
<i>C</i>		<i>R to 1</i>
<i>D</i>		<i>Read 1.307</i>

(b)

FIG. 36

Since $\cot = \frac{1}{\tan}$ the cotangent of any angle is found by a method similar to that described in Article 22 for the secant and cosecant. Thus for angles $5^\circ 43'$ to 45° with the slide in normal position set the angle A on T at the

reverse indicator and at the D index on C read $\cot A$. For angles 45° to 90° set $90^\circ - A$ on T at the reverse indicator and at the D index on C read $\cot A$. For angles less than $5^\circ 43'$ use the S and A scales exactly as described for sines in Article 22.

The use of the T scale in the trigonometric solution of triangles will be described in Chapter III.

(NOTE. — Many useful conversion settings are stated and the settings for a number of useful and typical problems are diagrammed for the standard Mannheim slide rule in Chapter IV.)

CHAPTER III

MODIFIED FORMS OF THE MANNHEIM RULE AND THEIR USE

A. THE POLYPHASE MANNHEIM SLIDE RULE

24. The Polyphase Mannheim Slide Rule. — A brief description of the method of using the inverted slide has been given in Article 20. The conveniences of the inverted slide were recognized early in the development of the slide rule, the earliest record of its use being that of William Hunt in 1697. The regular logarithmic scale was soon afterwards used in the inverted position but was not printed on the rule in the inverted position (to read normally from right to left) until a hundred years later, in 1797. The form finally standardized by Mannheim and treated at length in Chapter II does not carry an inverted scale.

An inverted C scale has, however, been incorporated in the Mannheim rule by a number of designers, the resulting rules being in form the same as the standard Mannheim with the inverted C scale on the slide between the regular B and C scales. Several such rules are supplied in the United States under the names multiplex,

maniphase, polyphase, etc., but all are essentially the same. One such form was produced by Keuffel & Esser about 1909 and is in very wide use.

Another scale which has been added to the Mannheim slide rule is a scale of cubes. It consists of three equal and similar parts laid out and divided in the same way as the A and B scales are formed. It is used in connection with the D scale for finding cubes and cube roots without setting the slide.

A Mannheim slide rule carrying both the inverted and cube scales is supplied by Keuffel & Esser and is called by them the "Polyphase (Mannheim) Slide Rule." This rule is illustrated in Fig. 37.



FIG. 37. POLYPHASE SLIDE RULE

The mechanical construction and form of the polyphase rule are the same as the standard Mannheim except that its width is about one eighth inch greater in order to provide space for the cube or K scale which is placed on the stock just below the D scale. (In some of the earlier forms of this rule the K scale is found on the edge of the stock instead of the face and the indicator is a line

or mark placed on the frame of the runner in line with the hair line of the runner.) The slide is of the same width as the slide of the Mannheim rule and bears the same scales on its reverse face (S, L and T).

25. The Inverted C Scale and its Use. — The inverted C scale is placed on the front face of the slide between the B and C scales and is marked CI. It runs from right to left with the indexes aligned with those of B and C and the graduations are exactly the same as those of C. The figures on the CI scale are usually printed in red while those on the other scales are in black.

The principle involved in the use of the CI scale is discussed in Articles 15 and 20. In the actual use of the inverted slide as there explained, however, are several disadvantages. Thus the scale must be read upside down, the B scale cannot be used in its regular manner, and there is no normal C scale. In the polyphase rule the regular arrangement and its advantages are not disturbed but to these are added all the advantages of the inverted scale. The same rules for decimal points apply as with the regular C scale or the decimal point is located by inspection.

With the slide in its normal position the reciprocal of a number is read on CI (or C) by setting the runner at the number on C (or CI). Similarly decimal fractions are

converted into common fractions with numerator 1 by setting the decimal fraction on C (or CI) and reading the denominator of the common fraction on CI (or C). Thus the reciprocal of 1.645 is $1 \div 1.645 = .608$. Similarly the decimal fraction .0555 is very nearly equal to the common fraction $\frac{1}{18}$ and the decimal fraction $.0532 = \frac{1}{18.2} = \frac{10}{182} = \frac{5}{91}$.

If the CI index is set on any number on D the product of every coinciding pair of numbers on CI and D, read at the indicator at the same setting, is equal to that number on D.

The operations of multiplication and division are interchanged when CI, D are used. Thus for *multiplication*:

- (1) To *Multiplicand* on D set *Multiplier* on CI,
- (2) To CI index set runner,
- (3) At indicator on D read Product;

while for *division*:

- (1) To *Dividend* on D set CI index,
- (2) To *Divisor* on CI set runner,
- (3) At indicator on D read Quotient.

Since the mechanical performance of these operations is familiar no examples or settings are given here.

In ordinary multiplication or division alone, however, the regular C, D scales are ordinarily used in the regular manner. The use of CI, D has the advantage, however, that when CI is set for the second factor the index is

automatically set and it is not necessary to make a choice of index as is the case when using C, D.

The greatest advantage of the CI scale lies in the multiplication of three factors, which will readily be understood from the following considerations. The product

$$a \times b \times c = X \quad (31)$$

may be written as

$$\left(a \div \frac{1}{b}\right) \times c = X. \quad (32)$$

This is the same as the combined multiplication and division of formula (24), Article 17, and so is simply set on the slide rule according to Fig. 26 (a). The only difference in the operations corresponding to formulas (24) and (32) is that in (24) C is used to divide by b while in (32) CI is used to divide by $\frac{1}{b}$. Thus the operation corresponding to formula (31), to which (32) is equivalent, is stated as follows:

- (1) To a on D set b on CI,
- (2) To c on C set runner,
- (3) At indicator on D read X .

This setting is diagrammed in Fig. 38 (a). It is much simpler than that shown and described for the same operation in Fig. 18 (c) in which only C, D are used.

An example of this operation is shown in Fig. 38 (b) which gives the setting for $25.7 \times 5.67 \times 4.12 = 600$.

<i>CI</i>	<i>Set b</i>	
<i>C</i>		<i>R to c</i>
<i>D</i>	<i>To a</i>	<i>Read X</i>

(a)

<i>CI</i>	<i>Set 5.67</i>	
<i>C</i>		<i>R to 4.12</i>
<i>D</i>	<i>To 2.57</i>	<i>Read 600</i>

(b)

FIG. 38

The method just described is readily extended to four or more factors. The formula

$$a \times b \times c \times d \times e = X \quad (33)$$

can be written in the form

$$a \div \frac{1}{b} \times c \div \frac{1}{d} \times e = X$$

or as

$$\frac{a \times c \times e}{\left(\frac{1}{b}\right) \times \left(\frac{1}{d}\right)} = X$$

and in this form is seen to be of the same form as formula (25), Article 17. The operation corresponding to formula (33) is therefore carried out as described in Article 17

for formula (25) except that b and d are set on CI instead of C.

If a certain number is to be divided in turn by each of a series of different numbers (multiplied by their reciprocals) the method described in Article 20 for the inverted slide is used, viz., set the index to the constant dividend and then set the runner in turn on each of the divisors on CI, reading the corresponding quotients on D. Of course, if a single number is to be multiplied in turn by each of a series of different numbers the slide index is set to the number on D, the runner to the multipliers on C in turn, and the products read on D.

Although a very convenient method of finding the cubes of numbers will be described in connection with the scale of cubes a more accurate method is available with the CI scale used in connection with C and D. Thus the cube of any number a is $a^3 = a \times a \times a$, or $a \times a \times a = X$. This operation is exactly the multiplication of three numbers as shown in Fig. 38 (b) with b and c replaced by a . This may be stated as follows:

- (1) To a on D set a on CI,
- (2) To a on C set runner,
- (3) At indicator on D read a^3 .

Inverse proportion is handled with the CI scale in the same manner as with the inverted slide using CI for C Inv.

The settings described above cover the basic direct operations with the CI scale. Many other settings in which CI is used in connection with other scales are given in Chapter IV.

26. The Cube Scale and Its Use. — It has been shown in Articles 10, 11, 19 how the A and B scales give the squares of numbers on D and C respectively. The principle of these scales is explained in connection with Figs. 4, 5, 6 and it is there shown that the drawing scale to which A is laid off is one half of that for D, or in other words the logarithm laid off on a given length of A is twice the logarithm laid off on the same length on D. Similarly if another scale were to represent three times the logarithm on the same length on D it would be expected that the numbers on the new scale would be the cubes of the numbers on D. This is indeed the case and the K scale on the polyphase slide rule is such a cube scale. Conversely of course the numbers on D are the cube roots of the corresponding numbers on K.

As stated in Article 24 the K scale is adjacent to D on the face of the stock of the polyphase slide rule. It has four indexes, the first and fourth of which are aligned with those of D. The two intermediate indexes divide K into three equal and exactly similar parts or scales which will be referred to as K1, K2, K3, reading from left to

right. Each part is divided and graduated exactly the same as the two parts of the A, B scales. Since each part is only two thirds the length of one half of A or B, however, readings cannot be made as closely on K as on A, B. When D is read as 1 to 10, then in order to give the cubes of D numbers K must be read as 1 to 1000 and the three parts as 1 to 10 to 100 to 1000. If D reads 10 to 100, K reads 1000 to 10,000 to 100,000 to 1,000,000; if D reads .1 to 1, K reads .001 to .01 to .1 to 1; etc. Thus when the decimal point is given in numbers on D the position of the decimal point in numbers on K is automatically fixed and at once known, and vice versa. Convenient rules will presently be given, however, which provide a ready means of pointing off cubes and cube roots of numbers as found on the K and D scales.

To find the cube of any number it is only necessary to set the runner at the number on D and its cube is at once read at the indicator on K.

To find the cube root of any number the runner is set at the number on K and its cube root is read at the indicator on D.

While there is only one place on D to set any particular number, however, there are three such places on K and these will give three different results for the cube root. Choice is made among these three by utilizing the method of cube root extraction of arithmetic in a manner analogous

to that in which the square root method of arithmetic was used in Article 19 to determine the A scale settings for square roots. Thus we have the

RULE: *For numbers greater than 1 begin at the decimal point and mark off the number into groups of three figures proceeding toward the left. If the last group contains one, two or three figures use K1, K2, K3 respectively.*

If the number is a decimal fraction use the following

RULE: *Mark off the fraction in groups of three figures beginning at the decimal point and including ciphers. If the first group containing digits after the ciphers contains one, two or three such digits use K1, K2, K3 respectively.*

The decimal points in cubes and cube roots are located by the following rules which apply to positive or negative characteristics (numbers greater than 1 or decimal fractions):

RULE: *If the cube of a number is found on K1, K2, K3, to find the characteristic of the cube multiply that of the number by 3 and subtract 2, 1, 0 respectively.*

RULE: *If the cube root of a number is found from K1, K2, K3, to find the characteristic of the root add 2, 1, 0 respectively to that of the number and divide the result by 3.*

Due to the smallness of the divisions of the K scale readings cannot be taken so accurately as on A, B and C. The method for extracting cube roots described in connection with those scales in Article 19 is therefore some-

times more useful. The most accurate method of all is that in which the full length C, CI and D scales are used, that is, the inverse operation of that described for cubes at the end of Article 25.

By using the K scale in connection with the other scales on the face of the rule any number may be raised to powers such as the following, either positive or negative:

$$\frac{2}{3}, \frac{3}{2}, 4, 5, 6, 9, \frac{5}{2}, \frac{5}{3}, \frac{5}{8}, \frac{4}{3}, \text{etc.}$$

and such expressions as the following may be evaluated at a single setting:

$$a \div \sqrt[3]{b^2}, a \div \sqrt{b^3}, a^2 \times \sqrt[3]{b^2}, \sqrt{a^3 \times b^3} \div c^3, \text{etc.}$$

Some of these operations will now be explained and described.

By formula (9) in Article 8, $a^{2/3} = \sqrt[3]{a^2}$ or $(\sqrt[3]{a})^2$. If the number a is set on K then $\sqrt[3]{a}$ is on D and the square of $\sqrt[3]{a}$ is at the indicator on A. Therefore to find $a^{2/3}$ set the runner at a on K and read $a^{2/3}$ at the indicator on A.

Since by formula (5) in Article 8, $a^{-2/3} = 1 \div a^{2/3}$, when $a^{2/3}$ is found, then this result is set on C and $a^{-2/3}$ is read at the indicator on CI. Similarly $a^{3/2} = (\sqrt{a})^3$ and when a is set on A, $a^{3/2}$ is read on K and $a^{-3/2}$ can then be found with C and CI.

Since by formula (1) $a^5 = a^3 \times a^2$ the fifth power is found at one setting in the following manner which is illustrative of the possible methods of combination of settings:

- (1) To a on D set runner and C index,
- (2) At indicator on K read a^3 and set runner to this on B,
- (3) At indicator on A read a^5 .

Thus a^2 on A is multiplied by a^3 on B as read from K. It is not necessary to read a^2 on A since when the C index is set at a on D the B index is thereby set at a^2 .

By formula (3) $a^6 = (a^3)^2 = (a^2)^3$, hence the sixth power is found as the square of the cube or the cube of the square. Similarly the fourth power is the square of the square and the ninth is the cube of the cube.

Other positive fractional powers are found by applying formula (9) with the K, A, D scales and the negative powers are the reciprocals of the positive powers and are found with C and CI. Thus

$$\begin{aligned} a^{5/2} &= \sqrt{a^5}, & a^{5/3} &= \sqrt[3]{a^5}, \\ a^{5/4} &= \sqrt{a^{5/2}}, & a^{5/6} &= \sqrt[3]{a^{5/2}}, \text{ etc.} \end{aligned}$$

The remaining expressions mentioned above together with many others are evaluated by means of settings stated and diagrammed in Chapter IV.

B. THE POLYPHASE DUPLEX SLIDE RULE

27. The Duplex Rule and the Folded Scales. —

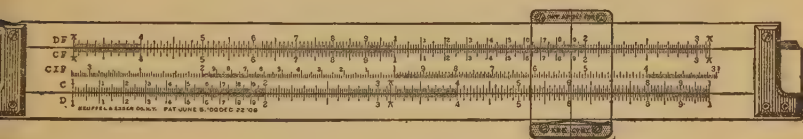
Although the slide of the Mannheim slide rule has scales graduated on both faces the stock is graduated on only one face. Slide rules having scales on both faces of the stock as well as the slide are called "duplex" rules. We

have seen that the first duplex slide rule was that produced by Seth Partridge in 1657. The form of duplex now in common use in the United States was brought out by William Cox in 1891 and uses the standard Mannheim scales. The slide moves between two outer strips of wood which are held together by thin metal cleats and all three strips are of the same thickness, so that their faces are flush and scales may be graduated on both faces of all three strips.

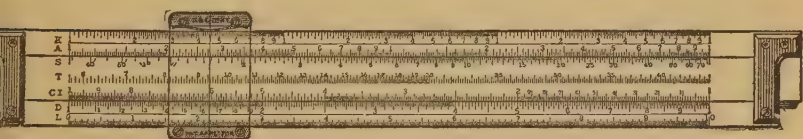
In 1817, as we have seen, an arrangement called the "folded scale" was introduced. In this arrangement the scale is "folded" near the center and the indexes placed together. The scale is then parted at the folding point and the separated ends are made the ends of the new scale, while the now coincident original indexes form a single index near the center of the new scale. This arrangement has been applied to the Mannheim scales by French manufacturers and by Keuffel & Esser in the United States. A duplex rule with the folded scales and all the regular scales of the polyphase Mannheim rule is supplied by Keuffel & Esser and is known as the "polyphase duplex" slide rule. This form of rule is widely used among engineers in the United States.

The form and scales of the polyphase duplex slide rule are shown in Fig. 39, in which (a) shows the main face or front of the rule and (b) the reverse face or back of the rule.

As seen in Fig. 39 (a), the front of the stock carries the regular D scale and the folded D scale, called DF; the front of the slide carries the regular and folded C scales, C and CF, and the folded CI scale, CIF. The back of the stock carries the regular L, D, A and K scales and the back of the slide the regular CI, T, S and B scales.



(a) Front



(b) Back

FIG. 39. POLYPHASE DUPLEX SLIDE RULE

The D, A and B scales are in their usual positions but L is now fixed adjacent to D with which it is used and CI is also adjacent to D with which it is used. The S scale is also fixed adjacent to its associated scale B or may be set in connection with A, and the T scale is conveniently placed with regard to D, with which it is used. On the front C and D are in their usual positions and immediately above are their associated scales CF and DF, D and DF being in a fixed relation on the stock while C and CF are

fixed on the slide. Similarly CF and its invert CIF are in a fixed relation on the slide, as are C and CI on the opposite face.

The runner extends around the entire rule and carries two indicator lines which are exactly aligned. With this arrangement not only can any two scales on the same face be read together, but also any two scales on opposite faces; thus any or all of the scales may be read together. This arrangement reduces the number of settings required in complicated operations and facilitates the use of the rule in general.

Instead of being folded at the center which is $\sqrt{10} = 3.1623$, the C, CI, D scales are folded at $\pi = 3.1416$ to give CF, CIF and DF. Thus the C and D indexes are aligned with π on CF and DF respectively and the middle indexes of CF, DF with π on C, D respectively. Now we have already seen that with the C index set at any number on D any two coinciding numbers on C, D are in the ratio equal to the number at which the C index is set. Therefore the ratio of all aligned numbers on DF and D and on CF and C is equal to π , and any number on DF or CF is π times its corresponding number on D or C respectively.

28. Operation of the Polyphase Duplex Slide Rule. — The A, B, C, D and L, S, T scales of the polyphase duplex rule are the regular Mannheim scales and their use is

precisely the same as described in Chapter II. The K and CI scales are the same as on the polyphase (Mannheim) rule and their use is described above in Articles 25 and 26. The use of these scales need not therefore be described here and will be referred to only in connection with the folded scales and the new combinations made possible by the rearrangement of their positions.

In the description of the polyphase duplex rule in the last Article (27) the multiplication of numbers by π is explained, viz., set the runner at the number on C or D and read π times the number on CF or DF respectively at the indicator. In the same way any number on CI is aligned with π times the number on CIF. These settings are illustrated in Fig. 39. Thus in (a) the indicator is set at 6 on D and at the indicator on DF is $6\pi = 18.85$. Also (since the slide and stock indexes are aligned in Fig. 39) 6 on C is aligned with $6\pi = 18.85$ on CF. Conversely there is found on C or D the quotient by π of any number on CF or DF respectively. Thus we may consider the indicator in Fig. 39 (a) as set at 18.85 on DF and on D find $18.85 \div \pi = 6$, and similarly on CF and C.

These scales furnish a very convenient table of diameters (d) and circumferences (c) of circles, for $c = \pi d$ and $d = \frac{c}{\pi}$. If the runner is set at any diameter on D the corresponding circumference is read at the indicator on

DF, and if it is set at any circumference on DF the corresponding diameter is read on D. C and CF may of course be used in the same way.

Any number which can be read on C or D in connection with any of the regular scales (products, quotients, cube roots, square roots, etc.) can be immediately multiplied by π by the use of the folded scales. Thus if any number a is set on K, then $\pi\sqrt[3]{a}$ is read at the indicator on DF directly, and if a is set on A then $\pi\sqrt{a}$ is read directly on DF. If $a \times b = x$ is set in the usual manner on C, D, then since x is on D, πx is at the same position of the indicator on DF. Since $\pi x = \pi ab$ we have the following setting for $\pi ab = X$:

- (1) To a on D set C index,
- (2) To b on C set runner,
- (3) At indicator on DF read X .

Similarly for the operation $\frac{\pi a}{b} = X$:

- (1) To a on D set b on C,
- (2) To C index set Runner,
- (3) At indicator on DF read X .

Similarly also such operations as

$$\frac{\pi ac}{b}, \frac{\pi ac}{bd}, \frac{\pi ace}{bd}, \text{ etc.,}$$

can be performed in the same general manner. Simply make the settings on C, D, CI in the usual manner as for

$$\frac{ac}{b}, \frac{ac}{bd}, \frac{ace}{bd}, \text{ etc.,}$$

and when the runner is brought to its final position read on DF instead of on D. This means that the result of the C, CI, D setting is multiplied by π .

Since CF and DF are exactly alike these two scales may be used for multiplication and division in the same manner as C and D or A and B are used together. It is only necessary to remember that the indexes of CF and of DF lie together near the centers of the scales instead of at the ends. Thus to carry out the multiplication $3 \times 6 = 18$, at 3 on DF set the (middle) CF index and at 6 on CF read the product 18 on DF. To perform the division $8.4 \div 4 = 2.1$, at 8.4 on DF set 4 on CF and at the (middle) CF index read the quotient 2.1 on DF.

The scale CIF is the reciprocal or invert of CF. Therefore in precisely the same manner as on C, D and CI the operations

$$\frac{a}{b}, \frac{ac}{b}, \frac{ac}{bd}, \frac{ace}{bd}, \text{etc.},$$

can be carried out on CF, DF and CIF and then, since any result on DF is divided by π on D, by making these settings on CF and DF and reading on D at the final position of the runner instead of on DF the results

$$\frac{a}{\pi b}, \frac{ac}{\pi b}, \frac{ac}{\pi bd}, \frac{ace}{\pi bd}, \text{etc.},$$

are obtained.

It can be said in general, therefore, that any operation

involving π as a factor can be carried out on scales C, D in the usual manner without π and the result read on DF instead of D at the final position of the runner; and vice versa any operation involving π as a divisor can be carried out on CF and DF in the usual manner without π and the final result read on D instead of DF.

A useful particular case of this general statement is that of reciprocals. Reciprocals of numbers are found on the polyphase duplex rule as on the polyphase Mannheim but the C and CI scales are on opposite faces of the rule and when a number is set on C the rule is turned over to read the reciprocal on CI. This operation can be performed even more simply on CF and CIF since these scales are reciprocal and are adjacent on the front of the rule. Also, since if the number a is set on C then πa is read on CF and the reciprocal of this quantity on CIF, the expression $\frac{1}{\pi a}$ is read directly on CIF when a is set on

C. Similarly if a is set on CF $\frac{\pi}{a}$ is read on CI. For, when a is on CF then $\frac{a}{\pi}$ is on C and $1 \div \left(\frac{a}{\pi}\right) = \frac{\pi}{a}$ is on CI.

Another very useful application of the folded scales (in reality the purpose for which the folded scales were designed) is illustrated by the following example.

$$\frac{52 \times 7}{13} = 28$$

The usual procedure is

- (1) To 52 on D set 13 on C,
- (i) (2) To 7 on C set runner,
- (3) At indicator on D read 28.

When the attempt is made to carry out step (3), however, it is found that the setting is "off the scale" and the usual procedure is to exchange C indexes and then go ahead with steps (2) and (3); so that the operation (i) really amounts to

- (1) To 52 on D set 13 on C,
- (2) To C index set runner,
- (ii) (3) To runner set other C index,
- (4) To 7 on C set runner,
- (5) At indicator on D read 28.

In this form the operation consists of five steps and is both inconvenient and liable to error. By the use of the folded scales, however, form (ii) can usually be avoided and form (i) used as if step (2) did not "run off the scale." Thus

- (1) To 52 on D set 13 on C,
- (iii) (2) To 7 on CF set runner,
- (3) At indicator on DF read 28.

For this form of the setting to be used it is generally (though not always) necessary for at least half of the length of the slide to be inside the stock. Thus the operation $\frac{52 \times 9}{13} = 36$ could not be carried out by either

of the methods (i) or (iii) but by using CF and DF first and then C and D it becomes simply

- (iv) (1) To 52 on DF set 13 on CF,
- (2) To 9 on C set runner,
- (3) At indicator on D read 36.

With practice any setting such as (i) or (ii) and their extensions can be replaced by one like (iii) or (iv) and much time saved. In general, in any operation in which a setting of the runner on C "runs off the scale" set it on CF instead and read on DF, and vice versa. It is easily seen that this is possible because the folded scales are graduated exactly like the regular scales but are so arranged that, in effect, the portion of the regular scale which projects beyond the reach of the runner is doubled back so as to be inside the stock.

The direct uses of the folded scales in connection with the standard (polyphase) scales have now been described. In the same way as for combination settings of the standard scales, so also combination settings of these with the folded scales can be multiplied indefinitely. A large number of such settings are given for the polyphase duplex slide rule in the next chapter.

C. THE LOG-LOG DUPLEX SLIDE RULE

In order properly to understand the so-called "log-log" slide rule it is necessary to extend the short discussion of exponents and logarithms given in Articles 8 and 9.

In general the common logarithms, or logarithms to base 10 are used and the standard slide rule scales (including inverted and folded scales) are based on those logarithms. In higher mathematics, however, and in steam, gas, hydraulic and electrical engineering the so-called "natural" logarithms (to base 2.7183) are used, together with expressions involving powers of this base called "exponentials." We will here state briefly some of the properties of natural logarithms and exponentials and their relations to common logarithms. As in Articles 8 and 9 the discussion will be more descriptive and explanatory than demonstrative and the reader who cares to study these subjects is referred to any good textbook of algebra or analytical trigonometry.

29. Natural Logarithms and Exponentials. — As implied in the definition of a logarithm in Article 9 logarithms may be referred to any number as base, the base 10 being used in ordinary calculation because of its convenience. In the theoretical investigations of higher mathematics and in the formulas of certain branches of physics and engineering it is more convenient to use the base mentioned above, which is designated by the letter " e ." (In certain work in electricity where " e " is used to indicate electromotive force the Greek letter "epsilon" (ϵ) is used.) This number is in reality, like π , an unending decimal

($e = 2.718281828 + \dots$) which makes its appearance in certain transformations in higher mathematics as the definite value to which the sum of the unending series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

becomes more nearly equal as the number of terms is continually increased. Logarithms referred to the base e are called "natural logarithms." Conversion of logarithms from the base e to the base 10 and 10 to e , or from any base to any other are readily made.

Thus let A be any number and let a and b be the bases of any two systems of logarithms. Then by the definition of a logarithm and by formula (14) of Article 9 we can write

$$\begin{aligned} m &= \log_b A, & n &= \log_a A, & (34) \\ b^m &= A, & a^n &= A, \\ a^n &= b^m. \end{aligned}$$

Taking the logarithm of both sides of this last equation to the base a we get, by formula (17),

$$n \log_a a = m \log_a b.$$

But by (23) $\log_a a = 1$, therefore this becomes

$$n = (\log_a b) \cdot m.$$

Putting in this the values of m and n from (34) gives finally

$$\log_a A = (\log_a b) \cdot \log_b A. \quad (35)$$

Hence if the logarithm of A to the base b is known and the logarithm of the same number A to some other base a

is desired it is only necessary to multiply the known logarithm by the logarithm of b to the base a . From (35) we have also

$$\log_b A = \left(\frac{1}{\log_a b} \right) \cdot \log_a A. \quad (36)$$

Thus suppose $\log_e A$ appears in a formula and no tables of logarithms to base e are available. By means of (36) $\log_{10} A$ may be used instead. In this case $a = 10$, $b = e = 2.7183$ and the formula (36) is

$$\log_e A = \left(\frac{1}{\log_{10} 2.7183} \right) \cdot \log_{10} A.$$

From a table of common logarithms it is found that the quantity $\log_{10} 2.7183 = 0.43429$; also $\frac{1}{.43429} = 2.3026$, and the conversion formula finally becomes

$$\log_e A = 2.3026 \log_{10} A. \quad (37)$$

Similarly (35) becomes

$$\log_{10} A = .43429 \log_e A. \quad (38)$$

For purposes of slide rule transformation (with logs to base 10 found on the L scale) the multiplier in (37) may be taken as 2.3; similarly the multiplier in (38) is easily remembered as .434.

By means of (37) 2.3 times the common logarithm is used in any formula for the natural logarithm; in this conversion both the characteristic and the mantissa of the common logarithm are multiplied by 2.3 to give the

natural logarithm. It is obvious that the characteristic of the natural logarithm thus found bears no simple relation to the number of digits in the given number A and is therefore not readily determined by inspection. For this reason tables of natural logarithms contain both mantissa and characteristic, the complete logarithm of every number in the table, with the decimal point properly placed. Such tables are given in handbooks of chemistry, physics and engineering.

The formulas of Article 9 apply to logarithms of any base (10, e , Napier's base, etc.). Thus from (23)

$$\log_e 1 = 0, \quad \log_e e = 1. \quad (39)$$

If the symbol ∞ is defined as a quantity infinitely great ("infinity")_then we may write as limiting values

$$\frac{1}{\infty} = 0, \quad b^{\infty} = \infty$$

and hence

$$\frac{1}{b^{\infty}} = 0, \quad \text{or} \quad b^{-\infty} = 0.$$

If we let b represent any base and take the logarithm of both sides of the last equation to the base b , then by formula (14)

$$\log_b 0 = -\infty$$

If $b = e$,

$$\log_e 0 = -\infty. \quad (40)$$

From the definitions of negative exponents and of logarithms it is obvious that the common logarithms of

decimal fractions (numbers between 0 and 1) are negative; it can be shown that this is the case with any base, in the same manner. This is illustrated by (40) and the first of formulas (39). Thus we have that all numbers above 1 have positive logarithms, the logarithm of 1 itself is zero, and in passing from 1 to zero (through all fractions) the logarithm goes from zero to minus infinity, that is, includes all negative numbers. Since the positive numbers 0 to $+\infty$ thus have all logarithms $-\infty$ to $+\infty$, negative numbers have no logarithms in the sense in which they have been defined here. Using the symbol $<$ to mean "less than" (and similarly $>$ would mean "greater than") then it can be said in general that if

$$0 < A < 1, \quad \log_e A < 0, \quad (41)$$

that is, negative. Of course this applies to any other base as well as to e .

If in the equation $3x + 2 = 8$, x is unknown the equation is called a "first degree" equation; $2x^2 - 7x + 5 = 0$ is a "second degree" equation. In each of these the unknown quantity is a base and the equation is described by reference to the exponent. Similarly such an equation as $4^x = 64$ in which the unknown quantity is the exponent is called an "exponential" equation and the quantity 4^x is the exponential. In particular are such expressions as e^x , e^{-x} , e^2 , etc., always known as "ex-

ponentials." Exponentials occupy a large place in the mathematics of physics and engineering.

The exponential equation

$$4^x = 64$$

is solved as follows: Take the logarithm (to any base) of both sides by (17),

$$\begin{aligned} x \log 4 &= \log 64 \\ \therefore x &= \frac{(\log 64)}{(\log 4)}. \end{aligned}$$

If the logarithms are common logarithms (as is the case in ordinary arithmetical calculations), then this expression gives

$$x = \frac{1.806}{.602} = 3$$

when the logarithm mantissas are read to three figures on the L scale of the slide rule. Of course it may be seen at once without using logarithms that when 4 to a certain power equals 64 then the power must be the cube, but in a case like

$$4.76^x = 2.19 \tag{42}$$

the value of x is not immediately obvious. Exactly as in the previous case, however,

$$x = \frac{(\log 2.19)}{(\log 4.76)} = \frac{.340}{.678} = .502; \tag{42a}$$

and in general if

$$b^x = a, \quad x = \frac{\log a}{\log b}. \tag{43}$$

As in the case of the base b so with the base e in equations (1) and (2) in Article 8,

$$e^x \times e^y = e^{x+y} \quad (44)$$

$$e^x \div e^y = e^{x-y}. \quad (45)$$

30. Exponentials, Powers and Roots, and the Slide Rule.—In order to carry out the solution of the exponential equation (42) as in (42a) by means of the slide rule the procedure is clearly as follows:

$$I \quad \begin{cases} \text{(i) Find } \log 2.19 = .340; \\ \text{(ii) Find } \log 4.76 = .678; \\ \text{(iii) Divide } \frac{.340}{.678} = .502. \end{cases}$$

These three operations (on Mannheim, polyphase or duplex rule) are:

- (i) $\begin{cases} \text{(1) To 219 on D set runner,} \\ \text{(2) At indicator on L read 340,} \\ \text{(3) Charact. of log is 0, write .340;} \end{cases}$
- (ii) $\begin{cases} \text{(1) To 476 on D set runner,} \\ \text{(2) At indicator on L read 678,} \\ \text{(3) Charact. of log is 0, write .678;} \end{cases}$
- (iii) $\begin{cases} \text{(1) To .340 on D set .678 on C,} \\ \text{(2) To C index set runner,} \\ \text{(3) At indicator on D read .502.} \end{cases}$

In the same way the solution of (43) can be carried out for any values of a and b .

Since in (43) a , b and x may have any values consistent with one another we can also write the similar equation

$$A^B = x \quad (46)$$

with x unknown and can also solve this equation logarithmically. Thus as an illustration let us consider

$$13.65^{2.37} = x. \quad (46a)$$

As before $2.37(\log 13.65) = \log x$. To find x then we must

$$\text{II} \quad \begin{cases} \text{(i)} & \text{Find } \log 13.65 = 1.135; \\ \text{(ii)} & \text{Multiply } 2.37 \times 1.135 = 2.69; \\ \text{(iii)} & \text{Find antilog } 2.69 = 186.0 \end{cases}$$

Each of these three operations is easily carried out on the C, D and L scales of the slide rule.

Consider next an equation of the form

$$x^a = B, \quad (47)$$

for example,

$$x^{3.62} = 79.5; \quad (47a)$$

As before

$$3.62(\log x) = \log 79.5;$$

$$\therefore \log x = \frac{(\log 79.5)}{3.62},$$

and the slide rule solution is

$$\text{III} \quad \begin{cases} \text{(i)} & \text{Find } \log 79.5 = 1.900; \\ \text{(ii)} & \text{Divide } \frac{1.900}{3.62} = .525; \\ \text{(iii)} & \text{Find antilog } .525 = 3.35. \end{cases}$$

The three operations (i), (ii), (iii) of each of the complete procedures II and III are separately simple and similar to those of I as described above and will not be given here in detail.

If the equation (47a) be compared with equations (18) and (7) it will be found that in solving (47a) we have

taken the 3.62th root of 79.5, that is, $x = \sqrt[3.62]{79.5} = 3.35$. Similarly in (46a) we have raised 13.65 to the 2.37th power, that is, $x = 13.65^{2.37} = 186.0$. In general, therefore, the slide rule can be used to find any power or root of any number by the ordinary logarithmic method, but as has appeared the operations require many settings. By means of the so-called "log-log" scale (or scales) used in connection with the ordinary Mannheim scales all three operations (i), (ii), (iii) of either procedure I, II or III can be performed at one setting.

Since equations (46) and (47) are only modifications of (43) a scale designed to solve (43) can be used to solve (46) and (47) in the same general way that C and D designed originally for multiplication can also be used for division and reciprocals. The log-log scale will therefore be described on the basis of equation (43) and then shown to apply to (46) and (47). It will also be seen that it furnishes a table of natural logarithms and exponentials.

31. The Log-Log Scale and the Log-Log Duplex Slide Rule. — We re-write here equations (43),

$$b^x = a, \quad x = \frac{(\log a)}{(\log b)}, \quad (43)$$

and for comparison also write equation (16) in the form

$$x = \frac{A}{B}, \quad \log x = \log A - \log B. \quad (16)$$

In order to find $x = \frac{A}{B}$ by means of the slide rule we have found that according to (16) two scales of logarithms of *numbers* are necessary and the difference between the lengths corresponding to $\log A$ and $\log B$ is the length corresponding to $\log x$, and the numbers being printed on the scales at the points corresponding to the scale lengths of their logarithms, A and B can be set on the scales and x read directly.

The scales necessary for the solution of (43) are now clearly defined: they must be laid off in lengths corresponding to logarithms of *logarithms*. Thus from (43)

$$\log x = \log(\log a) - \log(\log b) \quad (43a)$$

and if on a scale marked with logarithms of logarithms we find the difference between $\log(\log a)$ and $\log(\log b)$ and then lay off this distance on the ordinary scale of logarithms of numbers, this last distance will represent $\log x$ and the scale number will be x . The new scale of logarithms of *logarithms* is the "log-log" scale and the ordinary logarithmic scale of numbers is the regular Mannheim C scale. Something of the history of the origin and adoption of the log-log scale is given in Articles 4 and 7.

In construction the log-log scale is laid off in three or four sections on the stock of a duplex slide rule. This requires that the wooden strips forming the stock be

somewhat wider than in the polyphase duplex; otherwise the form is the same as that of the polyphase duplex rule. A complete rule formed in this manner is supplied by Keuffel & Esser and is called by them the "Log-log Duplex" slide rule. It is the same as the polyphase duplex rule of these manufacturers with the log-log scales added, as appears in Fig. 40, in which (a) shows the front and (b) the back of the complete rule.

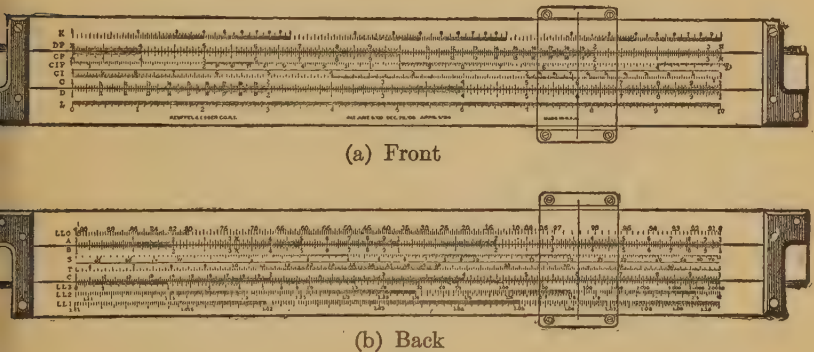


FIG. 40. LOG-LOG DUPLEX SLIDE RULE

As seen in Fig. 40 (a) the front of the stock of the log-log duplex slide rule carries regular and folded scales L, D, DF, K, and the front of the slide the C, CI, CIF, CF scales. In Fig. 40 (b) are seen on the back of the slide the standard B, S, T, C scales and on the back of the stock, the standard A scale and the log-log scale, marked LL and divided into four sections, LL0, LL1, LL2, LL3.

LL0 is above A and B in connection with which it is used, and LL1, LL2, LL3 are below C, in connection with which they are used, and their indexes are aligned with the right and left indexes of the stock.

The three LL sections, 1, 2, 3, if placed end to end would form a continuous scale from $e^{.01} = 1.01005$ to $e^{10} = 22026.3$. LL1, placed at the lower edge of the stock, reads from left to right $e^{.01} = 1.01005$ to $e^1 = 1.10517$; LL2 is just above LL1 and reads from left to right $e^1 = 1.10517$ to $e = 2.71828$; and LL3, contiguous to C, reads from left to right $e = 2.71828$ to $e^{10} = 22026.3$. Exponentials greater than e^{10} are rarely used so a section LL4 is not added. The right index of LL2 and the left index of LL3 are marked "e."

As shown in Article 29 above, equations (39), (40), (41), $\log_e 1 = 0$ and therefore $\log_{10}(\log_e 1) = -\infty$, and since $\log_{10}(\log_e 1.01) = -4.61015$, the difference between the log-log of 1.01 and of 1.00 is infinite and it would require a scale of infinite length to cover this range. LL1 therefore does not extend below log-log 1.01.

LL0 covers the range from $e^{-.03} = \frac{1}{e^{.03}} = .970$ to $e^{-3} = \frac{1}{e^3} = .050$ and is so arranged that e^{-1} is aligned with the right and left indexes of the A scale and $e^{-1} = \frac{1}{e} = .368$ is aligned with the middle A index.

The decimal point is already placed in all the numbers on LL and the scale divisions are decimal and self-explanatory.

The use of all scales on the log-log duplex slide rule other than LL has already been described in connection with the other rules and will not be discussed here except as they are used in connection with LL.

32. Use of the Log-Log Scales. — In order to explain the use of the log-log scales equations (43) and (43a) are here re-written for convenient reference:

$$b^x = a, \quad \log x = \log(\log a) - \log(\log b). \quad (48)$$

The method of finding x from the last of these equations by means of the logarithmic scales is shown in Fig. 41.

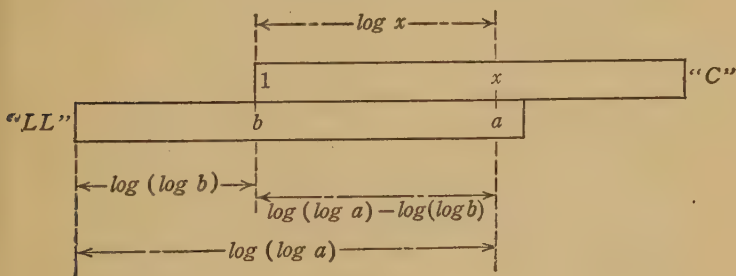


FIG. 41

On LL at a distance from the left index equal to $\log(\log b)$ is marked the number b , and at a distance from the left index equal to $\log(\log a)$ is marked the number a . The difference between these two lengths is $\log(\log a)$

$-\log(\log b)$. But according to equation (48) above, this is equal to $\log x$. Now the C scale is laid off in lengths equal to the logarithms of the numbers printed on it and at a distance from the left index on C equal to $\log x$ is marked the number x . If therefore the left index of C is set at b on LL as shown in Fig. 41 then x on C will be at a on LL. Thus to solve the exponential equation in (48), set the left C index at b on LL and at a on LL read x on C. As an example let us solve the simple equation already discussed:

$$4^x = 64.$$

The setting for this operation is stated as follows:

- (1) To 4 on LL set C index,
- (2) To 64 on LL set runner,
- (3) At indicator on C read 3.

This setting is diagrammed in Fig. 42.

<i>C</i>	<i>Set 1</i>	<i>Read $x=3$</i>
<i>LL3</i>	<i>To 4</i>	<i>R to 64</i>

FIG. 42

As has already been found in Article 30 the equation (46),

$$A^B = x, \tag{46}$$

is logarithmically of the same form as the equation $b^x = a$; the only difference lies in the fact that the unknown quantity x occupies a different position. Therefore

(46) is solved by the same type of setting as that just explained with the steps taken in a different order. It will here be explained independently, however. Applying the logarithmic formula (17) of Article 9 to (46) above we get

$$B(\log A) = (\log x)$$

and on applying to this the logarithmic formula for a product

$$\log B + \log(\log A) = \log(\log x) \quad (49)$$

or

$$\log B = \log(\log x) - \log(\log A).$$

This form shows that (46) is equivalent to the equation already solved and explained. For separate solution, however, it is used in the form (49). The setting for this solution is given in Fig. 43.

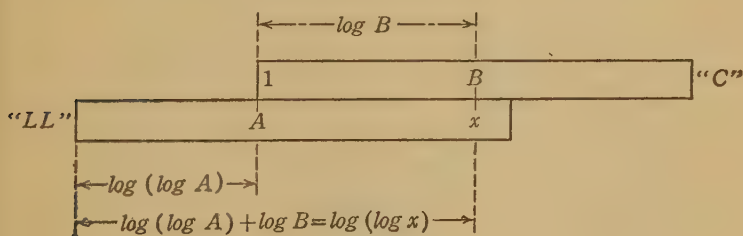


FIG. 43

On LL at a distance from the left index equal to $\log(\log A)$ is marked the number A, and on C at a distance from the left index equal to $\log B$ is marked the number B.

The sum of these two lengths is $\log(\log A) + \log B$. But according to (49) this is equal to $\log(\log x)$. If therefore the left index of C is set at A on LL as shown in Fig. 43, then x on LL will be at B on C. Thus to solve (49), which is (46), set the C index to A on LL and at B on C read x on LL. As an example consider the equation (46a) of Article 30, which has already been solved in three settings without the LL scale,

$$13.65^{2.37} = x. \quad (46a)$$

It is solved on the log-log rule in one setting as follows:

- (1) To 13.65 on LL set C index,
- (2) To 2.37 on C set runner,
- (3) At indicator on LL read 186.

This setting is diagrammed in Fig. 44.

C	Set 1	R to 2.37
LL3	To 13.65	Read 1.86

FIG. 44

Consider next equation (47),

$$x^A = B. \quad (47)$$

Taking logarithms as in the last case

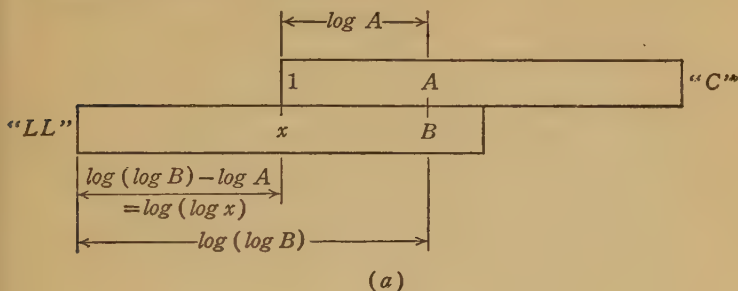
$$A(\log x) = (\log B),$$

and again taking logarithms,

$$\log A + \log(\log x) = \log(\log B),$$

$$\text{or} \quad \log(\log B) - \log A = \log(\log x), \quad (50)$$

which is logarithmically the same as (49) or (48). The solution is shown in Fig. 45 (a).



C	Set 3.62	R to 1
LL3	To 79.5	Read 3.35

(b)

FIG. 45

By tracing the steps through as was done above with Figs. 41 and 43 it is seen that A on C is set at B on LL and x is found on LL at the C index. As an example of this solution there is diagrammed in Fig. 45 (b) the setting for equation (47a) which is stated as follows:

- (1) To 79.5 on LL set 3.62 on C ,
- (2) To C index set runner,
- (3) At indicator on LL read 3.35.

For comparing and summarizing the above results and settings we write here the three equations (43), (46), (47):

$$b^x = a, \quad A^B = x, \quad x^A = B.$$

In this form it is seen that the three equations are all exponential in form with the unknown quantity x in a different position in each and with two known quantities in each, the first equation being the form ordinarily referred to as the "exponential equation." As we have seen above from the logarithmic forms these three equations all involve the same type of setting on the C and LL scales, one number, the exponent, being always read on C and the base and power on LL, the difference being that the steps are taken in different sequences in the three settings. If we take the A th root of the last equation it is seen at once that it may be written as $x = \sqrt[A]{B}$. The three equations are therefore

$$(i) \ x = A^B, \quad (ii) \ x = \sqrt[A]{B}, \quad (iii) \ b^x = a.$$

In this form it is seen that (i) represents the B th power of the number A ; (ii) represents the A th root of B ; and (iii) is an exponential equation.

In (i) A is a *number*, B is the *exponent* of the power, and x is the B th power.

In (ii) B is a *number*, A is the *index* of the root, and x is the A th root.

In (iii) b is the *base number*, x is the *exponential*, and a is the *power*.

These terms will be used below in stating rules for the slide rule calculation of x in (i), (ii), (iii).

Now a, b, A, B may be any numbers whatever; since all

three of the above equations have been solved in the settings explained and illustrated above, it is seen therefore that by means of the C and LL1, LL2, LL3 and the B and LL0 scales any root or power of any number (within the range of the scales) may be found, and exponential equations may be solved, at one setting of the slide and runner. Referring back to the settings corresponding to (i), (ii), (iii) in Figs. (43), (45a), (41) respectively, we have therefore the following log-log settings for powers, roots and exponentials:

(i) To find any POWER of any number:

- (1) To the *number* on LL set C index,
- (2) To the *power exponent* on C set runner,
- (3) At indicator on LL read the *power*.

(ii) To find any ROOT of any number:

- (1) To the *number* on LL set root *index* on C,
- (2) To C index set runner,
- (3) At indicator on LL read the *root*.

(iii) To solve an EXPONENTIAL equation:

- (1) To the *base* on LL set C index,
- (2) To the *power* on LL set runner,
- (3) At indicator on C read the *exponential*.

The numerical examples so far solved have required only one part of the log-log scale, LL3, and the C scale as used with LL3 has been used to read from 1 to 10, so that the numbers printed on C were units. Let us

now try to solve the exponential equation (42) in Article 29 by the setting (iii) above. Thus

- (1) To 4.76 on LL set C index,
- (2) To 2.19 on LL set runner,
- (3) At indicator on C read .502.

Step (1) is made at once but when the attempt is made to set the runner on 2.19 on LL, 2.19 is not found on LL3. Since LL3 begins at $e = 2.7183$ the number 2.19 is found on the next lower range or section, LL2. With the runner at 2.19 on LL2 the reading for x on C is 502 but if the step-by-step logarithmic solution in Article 29 is inspected it will be seen that the solution is .502 and not 5.02 as in our previous readings of C in connection with LL3.

Similarly let us find the fifth root of the number 1.25, that is, find $x = \sqrt[5]{1.25}$. According to setting (ii) above the root index 5 on C is set to 1.25 on LL2, the runner is then set on the C index and x is to be read at the indicator on LL. At the indicator on LL3, however, is 87 which is obviously not $\sqrt[5]{1.25}$. At the indicator on LL2 is 1.562 which also is obviously not $\sqrt[5]{1.25}$. If, however, the LL1 reading, 1.0456, is raised to the 5th power by the ordinary logarithmic method the result is 1.25. Therefore $x = \sqrt[5]{1.25} = 1.0456$ is to be read on LL1.

As a third example let us make the setting for $x = 2^7 = 128$. According to setting (i) above the (right) C index

is set at 2 on LL2 and the runner at 7 on C, and x is read at the indicator on LL. Since 2^7 is greater than 2 the result is to be sought farther along to the right on LL and when the end of LL2 is reached it is to be taken as continuing from 2.7183 at the left index of LL3. Thus 128 is found at the indicator on LL3.

A single setting will bring out the different relations of C to the three parts of LL. Thus let us raise 4 to the powers 5, .5, and .05. In accordance with setting (i) the C index is set at 4 on LL3 and the runner at the figure 5 on C. Now $4^5 = 1024$, $4^{.5} = 4^{1/2} = \sqrt{4} = 2$, $4^{.05} = 4^{1/20} = \sqrt[20]{4} = 1.0718$. With the runner at the figure 5 on C the numbers read at the indicator on LL (according to (3) of setting (i) above) are 1024 on LL3, 2 on LL2, and 1.0718 on LL1.

The results of the four examples just discussed indicate that the numbers on C are read as units in connection with LL3, they are read as tenths in connection with LL2, and as hundredths in connection with LL1. This might have been stated directly without examination by recalling the limits and range of each of the sections of LL. Thus LL1 runs from the .01 to the .10 powers of e (the hundredths); LL2 from 0.1 to 1.0 (tenths) powers; and LL3 1 to 10 (units) powers. Also, as found in the preceding examples, when a reading cannot be found on that section of LL on which the setting was begun it will be found on

a higher or lower section which is to be considered as a direct continuation of the complete LL scale. Since the decimal point is given on LL its position on C in any setting of C with LL will be obvious on inspection of the example.

Summarizing the preceding examples and the discussion of the last paragraph we can make the general statement (refer also to Article 31 and Fig. 40): LL1, LL2, LL3 read from $e^{.01}$ to $e^{.1}$ to e^1 to e^{10} and when slide and stock indexes are aligned and C is referred to LL3, LL2, LL1 the numbers on C are units, tenths, hundredths, respectively.

From this statement it is seen at once that there is another way of viewing the LL scale. Thus with slide and stock indexes aligned, 1 on C is at e^1 on LL3 and 10 on C is at e^{10} on LL3; similarly 4 on C is at e^4 on LL3, and so for any other number on C: that power of e is at that point on LL3. Similarly .1 on C is at $e^{.1}$ on LL2 and 1 on C is at e^1 on LL2, with corresponding relations for the intermediate numbers, and for the LL1 indexes and intermediate values. Thus if the runner is set at any number from .01 to .1 to 1 to 10 on C that power of e is read at the indicator on the corresponding section of LL with the decimal point placed. The LL scale, therefore, furnishes a table of exponentials such as is usually given in engineering and mathematical handbooks. Sections 1, 2, 3 of LL cover the range $e^{.01}$ to e^{10} of such a table and

the negative exponentials e^{-1} to e^{-3} are given on LL0 which is used in connection with A or B in the same manner as the other three sections are used with C.

The further use of LL0 in connection with A will be understood from the following considerations. LL0 covers the range $e^{-0.3} = .9703$ to $e^{-3} = .0498$, as stated in Article 31 (to three figures .0498 was there given as .050). $e^{-1} = .9048$ is at both the left and right indexes and $e^{-1} = .3679$ at the middle index, while both $e^{-.03}$ and e^{-3} are at the point 3 on A2. The figure 3 on A2 is therefore the beginning and the end of LL0. When A is read in direct connection with LL0 therefore the figure 3 on A2 is to be read as .03 at the beginning of LL0 and the remaining figures on A2 as .04 to .1; A1 is to be read as .1 to 1, and the first part of A2 as 1 to 3. A minus sign (−) is to be placed before each of these A numbers because the exponentials here are negative.

The complete table of exponentials furnished by LL in connection with A and C therefore covers the range $e^{-0.3}$ to e^{-3} and $e^{0.1}$ to e^{10} , which is ample for all practical purposes.

Now in accordance with the definition of logarithms to any base in Article 9 and of natural logarithms (base e) in Article 29 the exponents in the table of exponentials are nothing more than the natural logarithms of the numbers corresponding to them. Thus $e^{10} = 22026.3$

and $\log_e 22026.3 = 10$ according to equations (14). Similarly $e^{-.03} = .9703$ and $\log_e .9703 = - .03$, which is in accordance with formula (41), decimal fractions having negative logarithms. Therefore if the runner is set at any number on LL its natural logarithm is found at the indicator on A or C (indexes being aligned). The characteristic of the natural logarithm is included in the A or C scale reading and the decimal point is placed as already explained in connection with the reading of those scales with LL. The LL scale (complete) when thus used in connection with A and C furnishes a direct reading table of natural logarithms of numbers from .0498 to .9705 and 1.0101 to 22026. The gap 0.9705 to 1.0101 exists because $\log_e 1 = 0$ and the A and C scales do not extend to zero.

By making use of the logarithm conversion formulas (35) and (36) the logarithm of numbers to any base can be found by setting the C index to the *base* on LL; if the runner is then set to any number on LL its logarithm to the chosen base is read at the indicator on C. The reading so found includes the characteristic and the decimal point is to be placed in accordance with the rules of reading given above. In this way the common logarithms can be found by setting the C index to 10 on LL. Since scale C can ordinarily be read to three figures, however, while the mantissa of the common logarithm can be read to three

figures on scale L, it is better to read common logarithms on L and add the characteristic by the usual rule.

Numbers read on A, C and LL in any of the methods discussed in this article may of course then be used in connection with any of the other scales on the log-log slide rule as already explained in connection with the use of those scales on the Mannheim, polyphase and duplex rules. These various uses will not be described here but a large number of typical settings for the various scales of the log-log duplex rule are given in the next chapter.

D. THE SLIDE RULE IN TRIGONOMETRY

33. Introductory. — In Articles 22 and 23 the S and T scales of the slide rule are described and their use in finding the trigonometric functions of angles is explained, as is their use in multiplying other numbers by these functions. By applying these operations to the sides and angles of triangles all the triangle solutions of plane trigonometry may be carried out on the slide rule. With the exception of one or two cases no single computation in any of these solutions requires more than one setting of slide and runner.

In the following articles the operations involved in these solutions will be described and explained. In order to perform these operations on the Mannheim and polyphase slide rules it is necessary to reverse (but not invert) the

slide so as to have the S scale contiguous to A and the T scale to D. When this is done these four scales may by means of the runner be used together in the same manner on all slide rules; the descriptions given below apply therefore to all the forms already described.

In Article 19 two special gauge points (π and $\frac{1}{4}\pi$) on the A and B scales were described and their uses explained there and elsewhere. A few operations involving these gauge points which are useful in trigonometry will be given in the next article.

34. Radian and Degree Angle Measure Conversion. —

Since π radians equal 180 degrees, if any angle be expressed in both radians and degrees then $\pi : 180 :: (\text{no. rad.}) : (\text{no. deg.})$, or

$$\frac{\pi}{180} = \frac{\text{radians (A)}}{\text{degrees (B)}} \quad (51)$$

the capital letters at the right in parentheses indicating the scales on which the proportion is to be set up. Therefore to convert degrees to radians or radians to degrees we have the following operation:

- $$\left\{ \begin{array}{l} (1) \text{ To } \pi \text{ on A set 180 on B, and} \\ (2) \text{ At radians on A read degrees on B, or} \\ (3) \text{ At degrees on B read radians on A.} \end{array} \right.$$

If it is desired to convert half a given angle from radians to degrees the formula

$$\frac{\pi}{90} = \frac{\text{radians in angle (A)}}{\text{degrees in half angle (B)}} \quad (52)$$

is used. It is set up on A, B as indicated. In order to convert an angle in degrees to half the angle in radians the formula is:

$$\frac{\pi}{360} = \frac{\text{radians in half angle}}{\text{degrees in angle}} \quad \begin{matrix} \text{(A)} \\ \text{(B)} \end{matrix} \quad (53)$$

The above conversions can be more accurately but less conveniently carried out on the C, D scales by making use of the following three corresponding proportions and setting them up on C, D in the usual manner of proportions:

$$\frac{134}{2.34} = \frac{\text{degrees}}{\text{radians}} \quad (51a)$$

$$\frac{134}{4.68} = \frac{\text{degrees in half angle}}{\text{radians in angle}} \quad (52a)$$

$$\frac{134}{1.17} = \frac{\text{degrees in angle}}{\text{radians in half angle}} \quad (53a)$$

35. Solution of Right Triangles. — If the standard notation for right triangles given in Fig. 46 is used the fundamental formulas are:

$$\frac{a}{c} = \sin A = \cos B$$

$$\frac{b}{c} = \cos A = \sin B$$

$$\frac{a}{b} = \tan A = \cot B$$

also,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Since the angle C is in every case equal to 90° it is not necessary to state it in every solution.

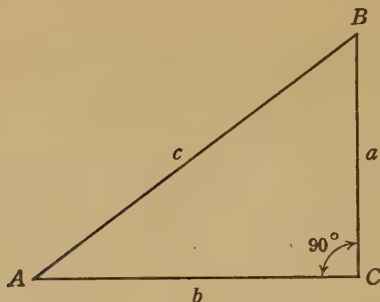


FIG. 46

For purposes of slide rule settings it is convenient to write the fundamental formulas in the form of proportions.

Thus the first one above may be written $\frac{a}{\sin A} = \frac{c}{1}$ but

$1 = \sin 90^\circ$. Therefore $\frac{a}{\sin A} = \frac{c}{\sin 90^\circ}$, which may be set

up on the A, S scales exactly as written. Similarly the

third of the above formulas may be written $\frac{\tan A}{a} = \frac{\tan 45}{b}$

since $\tan 45^\circ = 1$, and in this form the proportion may be set up directly on the T, D scales. To find either angle A or B when the other is known the relation $A + B = 90^\circ$ is used and the known angle is subtracted from 90° to give the other.

In each case of triangle solution the two given parts,

the three required parts, the appropriate solution formulas and the slide rule setting will be stated and a numerical example will be diagrammed. The decimal point will be placed as explained in Articles 22 and 23, or to correspond with that in the corresponding member of the solution proportion.

If in any case it is required to use the tangent of an angle less than $5^{\circ} 43'$ the S scale may be used instead of T as explained in Article 23.

CASE I. — Given two legs a, b ; to find A, B, c .

$$\frac{1}{b} = \frac{\tan A}{a}, \quad B = 90 - A, \quad \frac{a}{\sin A} = \frac{c}{1}.$$

$$(i) \quad \begin{cases} (1) \text{ To } b \text{ on D set } 45^{\circ} \text{ on T,} \\ (2) \text{ To } a \text{ on D set runner,} \\ (3) \text{ At indicator on T read } A. \end{cases}$$

$$(ii) \quad \begin{cases} (1) \text{ To } a \text{ on A set } B \text{ on S,} \\ (2) \text{ To } 90 \text{ on S set runner,} \\ (3) \text{ At indicator on A read } c. \end{cases}$$

Example: $a = .0227, b = .084$

T	Set 45°	A = $15^{\circ} 7'$	A	To .0227	c = .087
D	To 84	R to .0227	S	Set $15^{\circ} 7'$	R to 90°
(i)			(ii)		

$$B = 90^{\circ} - 15^{\circ} 7' = 74^{\circ} 53'$$

As an example of the use of S instead of T let $a = 64.4$, $b = 5.99$, find B by setting (i)

A	To 64.4	R to 5.99
S	Set 90°	B = $5^{\circ} 19'$
(i)		

CASE II. — Given leg and hypotenuse b, c ; to find a, A, B .

$$\frac{c}{1} = \frac{b}{\sin B}, \quad A = 90 - B, \quad \frac{b}{\sin B} = \frac{a}{\sin A}$$

- $$\left\{ \begin{array}{l} (1) \text{ To } c \text{ on A set 90 on S} \\ (2) \text{ To } b \text{ on A set runner} \\ (3) \text{ At indicator on S read } B; A = 90 - B, \text{ then} \\ (4) \text{ To } A \text{ on S set runner} \\ (5) \text{ At indicator on A read } a \end{array} \right.$$

Example: $b = .647, c = 7.46$.

A	To 7.46	R to .647		$a = 7.43$
S	Set 90°	$B = 4^\circ 59'$		$R \text{ to } (90 - 4^\circ 59') = 85^\circ 1'$

CASE III. — Given leg and adjacent angle b, A ; to find B, c, a .

$$B = 90^\circ - A, \quad \frac{b}{\sin B} = \frac{c}{1} = \frac{a}{\sin A}$$

After subtracting A from 90° to find B then

- $$\left\{ \begin{array}{l} (1) \text{ To } b \text{ on A set } B \text{ on S} \\ (2) \text{ To } 90 \text{ on S set runner} \\ (3) \text{ At indicator on A read } c; \text{ also} \\ (4) \text{ To } A \text{ on S set runner} \\ (5) \text{ At indicator on A read } a \end{array} \right.$$

Example: $b = .084, A = 15^\circ 9', B = 90^\circ - 15^\circ 9' = 74^\circ 51'$

A	To .084	$c = .087$		$a = .0227$
S	Set $74^\circ 51'$	$R \text{ to } 90^\circ$		$R \text{ to } 15^\circ 9'$

CASE IV. — Given a leg and opposite angle: a, A ;
to find B, b, c .

$$B = 90^\circ - A, \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1}$$

After subtracting A from 90° to find B , then

- (1) To a on A set A on S
- (2) To B on S set runner
- (3) At indicator on A read b ; also
- (4) To 90° on S set runner
- (5) At indicator on A read c .

Example: $a = 3.73, A = 24^\circ 20'.$ $B = 90^\circ - 24^\circ 20'$
 $= 65^\circ 40'$

A	To 3.73	$b = 8.25$	$c = 9.05$
S	Set $24^\circ 20'$	R to $65^\circ 40'$	R to 90°

CASE V. — Given hypotenuse and an acute angle:
 c, A ; to find B, a, b

$$B = 90^\circ - A, \quad \frac{c}{1} = \frac{a}{\sin A} = \frac{b}{\sin B}$$

After subtracting A from 90° to find B , then

- (1) To c on A set 90° on S
- (2) To A on B set runner
- (3) At indicator on A read a ; also
- (4) To B on S set runner
- (5) At indicator on A read b .

Example: $c = 426, A = 34^\circ 15'.$ $B = 90^\circ - 34^\circ 15'$
 $= 55^\circ 45'$

A	To 426	$a = 240$	$b = 352$
S	Set 90°	R to $34^\circ 15'$	R to $55^\circ 45'$

If it is desired to read b in Case V, c in Case IV, a in Case III and a in Case II more accurately but with an extra setting the tangent formula may be used and the setting made on the T and D scales. Thus in the last example, Case V, $\frac{\tan A}{a} = \frac{1}{b}$. The setting is below.

T	Set $34^{\circ} 15'$	R to 45°
D	To 240	$b = 352$

36. Solution of Oblique Triangles. — Using the notation of Fig. 47 the direct solution formulas for the oblique triangle are

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

$$\frac{\tan[\frac{1}{2}(A - B)]}{a - b} = \frac{\tan[\frac{1}{2}(A + B)]}{a + b} = \frac{\cot \frac{1}{2} C}{a + b},$$

with corresponding formulas for angles A , C and B , C .

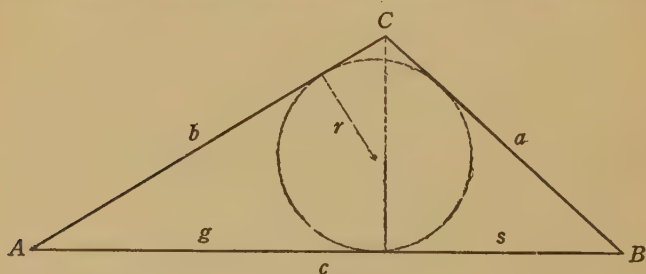


FIG. 47

When the sides are given two indirect methods may be used. One requires finding the radius r of the inscribed

circle and the other requires finding the two segments g , s into which the longest side is divided by the perpendicular from the opposite vertex. In the first method when

$$S = \frac{1}{2}(a + b + c), \quad r = \sqrt{\frac{(S - a)(S - b)(S - c)}{S}}$$

$$\frac{\tan \frac{1}{2} A}{r} = \frac{1}{S - a}, \quad \frac{\tan \frac{1}{2} B}{r} = \frac{1}{S - b}, \quad C = 180^\circ - (A + B).$$

In the second method (when the lengths are in the order of Fig. 47)

$$d = \frac{(b + a)(b - a)}{c}, \quad g = \frac{1}{2}(c + d), \quad s = \frac{1}{2}(c - d),$$

$$\frac{b}{1} = \frac{g}{\sin(90^\circ - A)}, \quad \frac{a}{1} = \frac{s}{\sin(90^\circ - B)}, \quad C = 180^\circ - (A + B).$$

As in the solution of right triangles, there are five cases of oblique triangles to consider. The difference lies in the fact that in right triangles one angle is always known, which is not the case in oblique triangles.

CASE I. — Given a side and two adjacent angles: a, B, C ; to find A, b, c .

$$A = 180^\circ - (B + C), \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

After finding A by subtracting $(B + C)$ from 180° , then

- (1) To a on A set A on S ,
- (2) At B on S read b on A ,
- (3) At C on S read c on A .

Example: $a = 25, B = 50^\circ, C = 60^\circ;$

$$A = 180^\circ - (50^\circ + 60^\circ) = 70^\circ.$$

$\overline{A}$	To 25	$b = 20.6$	$c = 23.2$
S	Set 70°	R to 50°	R to 60°

CASE II. — Given two angles and a side opposite one of them: $A, B, a;$ to find $C, b, c.$

$$C = 180^\circ - (A + B), \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

After subtracting $(A + B)$ from 180° to find C the setting is the same as in Case I.

CASE III. — Given two sides and an angle opposite one of them: $a, b, A;$ to find $C, B, c.$

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad C = 180^\circ - (A + B), \quad \frac{a}{\sin A} = \frac{c}{\sin C}.$$

(1) To a on A set A on S,

(2) At b on A read B on S; then $C = 180^\circ - (A + B),$
and

(3) At C on S read c on A.

If A is acute and $a > b$ there is one solution; otherwise there are two solutions.

Example 1: One solution: $A = 32^\circ 43', a = .988,$
 $b = .672.$

$\overline{A}$	To .988	R to .672	$c = 1.480$
S	Set $32^\circ 43'$	$B = 21^\circ 34'$	R to $54^\circ 17'$

Example 2: Two solutions: $A = 48^\circ 34'$, $a = 46.24$,
 $b = 60$.

A	To 46.24	R to 60	$c_1 = 50.4$	$c_2 = 29.1$
S	Set $48^\circ 34'$	$B_1 = 76^\circ 42'$ $B_2 = 180^\circ - B$ $= 103^\circ 18'$	R to $C_1 = 54^\circ 44'$ $= 180^\circ - (A + B_1)$	R to $C_2 = 28^\circ 8'$ $= 180^\circ - (A + B_2)$

CASE IV. — Given two sides and the included angle:
 A, b, c ; to find, B, C, a .

$$\frac{1}{2}(B + C) = 90^\circ - \frac{1}{2}A, \quad \frac{\tan[\frac{1}{2}(B + C)]}{b + c} = \frac{\tan[\frac{1}{2}(B - C)]}{b - c},$$

$$B = [\frac{1}{2}(B + C)] + [\frac{1}{2}(B - C)], \quad C = [\frac{1}{2}(B + C)] - [\frac{1}{2}(B - C)],$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

- (i) $\left\{ \begin{array}{l} (1) \text{ To } (b + c) \text{ on D set } \frac{1}{2}(B + C) \text{ on T,} \\ (2) \text{ To } (b - c) \text{ on D set runner,} \\ (3) \text{ At indicator on T read } \frac{1}{2}(B - C). \end{array} \right.$

Then after finding B on C from the formulas,

- (ii) $\left\{ \begin{array}{l} (1) \text{ To } b \text{ on A set } B \text{ on S,} \\ (2) \text{ At } A \text{ on S read } a \text{ on A.} \end{array} \right.$

If A is obtuse operation (i) requires a single setting; if A is acute a double setting is required

Example 1: Single setting: $A = 110^\circ 32'$, $b = 6.24$,
 $c = 2.35$.

$$[\frac{1}{2}(B + C)] = 90^\circ - 55^\circ 16' = 34^\circ 44', \quad b + c = 8.59,$$

$$b - c = 3.89$$

(i)	T	Set $34^\circ 44'$	$[\frac{1}{2}(B - C)] = 17^\circ 26'$
	D	To 8.59	R to 3.89

$$B = 34^\circ 44' + 17^\circ 26'$$

$$= 52^\circ 10'$$

$$C = 34^\circ 44' - 17^\circ 26'$$

$$= 17^\circ 18'$$

(ii)	A	To 6.24	$a = 7.39$
	S	Set $52^{\circ} 10'$	R to $180^{\circ} - 110^{\circ} 32'$ $= 69^{\circ} 28'$

Example 2: Double setting: $A = 59^{\circ} 13'$, $b = 226.2$,
 $c = 138.7$

$$[\frac{1}{2}(B + C)] = 90^{\circ} - 29^{\circ} 37' = 60^{\circ} 23', \quad b + c = 364.9,$$

$$b - c = 87.5$$

(i)	T	Set 45°	R to $29^{\circ} 37'$	Index to R	$[\frac{1}{2}(B - C)] = 22^{\circ} 54'$
	D	To 364.9			R to 87.5

$$B = 60^{\circ} 23' + 22^{\circ} 54'$$

$$= 83^{\circ} 17'$$

$$C = 60^{\circ} 23' - 22^{\circ} 54'$$

$$= 37^{\circ} 29'$$

(ii)	A	To 226	$a = 196$
	S	Set $83^{\circ} 17'$	R to $59^{\circ} 13'$

CASE V. — Given the three sides: a, b, c ; to find A, B, C :

Method 1. — $S = \frac{1}{2}(a+b+c)$, $r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}}$

$$\frac{1}{S-a} = \frac{\tan \frac{1}{2} A}{r}, \quad \frac{1}{S-b} = \frac{\tan \frac{1}{2} B}{r}, \quad C = 180^{\circ} - (A + B)$$

- (i) $\left\{ \begin{array}{l} (1) \text{ To } (S-a) \text{ on A set } S \text{ on B,} \\ (2) \text{ To } (S-b) \text{ on B set runner,} \\ (3) \text{ To runner set B index,} \\ (4) \text{ To } (S-c) \text{ on B set runner,} \\ (5) \text{ At indicator on D read } r. \end{array} \right.$

- (ii) $\left\{ \begin{array}{l} (1) \text{ To } (S-a) \text{ on D set } 45^{\circ} \text{ on T,} \\ (2) \text{ At } r \text{ on D read } \frac{1}{2} A \text{ on T.} \end{array} \right.$

- (iii) $\left\{ \begin{array}{l} (1) \text{ To } (S-b) \text{ on D set } 45^{\circ} \text{ on T,} \\ (2) \text{ At } r \text{ on D read } \frac{1}{2} B \text{ on T.} \end{array} \right.$

From 180° subtract $(A + B)$ to find C .

Example: $a = 260, b = 280, c = 300; S = 420, S - a = 160, S - b = 140, S - c = 120.$

(i)	A	To 160			
	B	Set 420	R to 140	Index to R	R to 120
	D				$r = 8$

(ii)	T	Set left I. $\frac{1}{2} A = 26^\circ 35'$		(iii)	T	Set L.I. $\frac{1}{2} B = 29^\circ 45'$	
	D	To 160	R to 8		D	To 140	R to 8

$A = 53^\circ 10', \quad B = 59^\circ 30', \quad C = 180^\circ - 112^\circ 40' = 67^\circ 20'$

Method 2. — $d = \frac{(b+a)(b-a)}{c}, g = \frac{1}{2}(c+d), s = \frac{1}{2}(c-d)$

$$\frac{b}{1} = \frac{g}{\sin(90^\circ - A)}, \quad \frac{a}{1} = \frac{s}{\sin(90^\circ - B)},$$

$$C = 180^\circ - (A + B).$$

- (i) $\begin{cases} (1) \text{ To } (b + a) \text{ on D set } c \text{ on C,} \\ (2) \text{ At } (b - a) \text{ on C read } d \text{ on D.} \end{cases}$
- (ii) $\begin{cases} (1) \text{ To } b \text{ on A set } 90^\circ \text{ on S,} \\ (2) \text{ At } g \text{ on A read } (90^\circ - A) \text{ on S.} \end{cases}$
- (iii) $\begin{cases} (1) \text{ To } a \text{ on A set } 90^\circ \text{ on S,} \\ (2) \text{ At } s \text{ on A read } (90^\circ - B) \text{ on S.} \end{cases}$

Example: $a = 260, b = 280, c = 300, (b + a) = 540, (b - a) = 20.$

(i)	C	Set 300	R to 20	$g = \frac{1}{2}(300 + 36) = 168$
	D	To 540	$d = 36$	$s = \frac{1}{2}(300 - 36) = 132$

A	To 280	R to 168	A	To 260	R to 132
S	Set 90°	$(90^\circ - A) = 36^\circ 50'$	S	Set 90°	$(90^\circ - B) = 30^\circ 30'$
	(ii)			(iii)	$q\frac{1}{2}$

$A = 53^\circ 10', \quad B = 59^\circ 30', \quad C = 180^\circ - 112^\circ 40' = 67^\circ 20'$

This completes the solution of all cases of plane triangles. Problems in spherical trigonometry will not be treated here.

37. Slide Rule Check of Logarithmic Solutions. — So far we have only considered the slide rule as performing original calculations. It is, of course, also very useful in checking calculations made more accurately by other methods, such as "long hand" arithmetical calculations or logarithmic computations, and as an aid in carrying out logarithmic computations themselves. One of the most useful and convenient of these checks is that of triangle solutions carried out with logarithm tables.

In any plane triangle, right or oblique, the so-called sine law must hold:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

where A, B, C are the angles and a, b, c the corresponding opposite sides. In the usual notation (as used in Article 35) for right triangles C is 90° but the equations still apply. If therefore values of all the six parts of any triangle are known and any angle is set on the S scale at its opposite side on A, then each of the other two angles on S will be at its opposite side on A. Therefore to check any logarithmic solution or graphic solution of a triangle set any one of the angles obtained on S to its side on A.

If the solution is correct then each of the other two angles on S will be at its opposite side on A, thus:

$$\begin{array}{c|c|c|c} A & a & b & c \\ \hline S & A & B & C \end{array}$$

If they are not so found, the solution is not correct; if they are so found, the solution is correct at least to the first few figures of each side and to within a very few minutes of each angle.

38. Interpolation of Logarithms.— If a table of logarithms is not provided with auxiliary tables of proportional parts interpolation between tabulated values is tedious and liable to error. If such auxiliary tables are provided they frequently require interpolation in order to obtain the last figure given by the table. The slide rule is simply and easily used to replace the table of proportional parts in the first case and to extend or supplement it in the second case.

When the differences between successive tabulated values are small the number differences and the corresponding logarithm differences are proportional. Therefore when the differences between any two consecutive numbers and between their logarithms are known a difference in one corresponding to an intermediate difference in the other is found by solving a proportion with three

known members. The slide rule is therefore especially adapted to finding the required difference.

Let n and N be two consecutive numbers in a table of logarithms, and l and L their respective logarithms. Let D be the logarithm difference ($L - l$); the corresponding number difference is $(N - n) = 1$. If then d is a fractional difference intermediate between n and N , and X is the corresponding logarithm difference, we have

$$\frac{1}{D} = \frac{d}{X}, \quad \text{or} \quad X = Dd. \quad (51)$$

As in every case, therefore, the solution may be performed as a multiplication or as a proportion. In either case it is stated as follows:

- (1) To *total log diff.* on D set C index,
- (2) To *partial number diff.* on C set runner,
- (3) At indicator on D read *partial log. diff.*

As an example, find in a six-place table the common logarithm of 7935.73. From the table

$$\begin{array}{l} \log 7936 = 3.899602 \\ \log 7935 = 3.899547 \end{array}$$

$$\begin{array}{l} D = 55 \text{ (in last two places)} \\ \text{For } 7935.73, d = .73 \\ X = 40.1 \text{ (in last two places)} \\ \therefore \log 7935.73 = 3.899587 \end{array}$$

The slide rule setting for the interpolation is:

C	Set 1	R to .73
D	To 55	X = 40.1

The same form of setting holds for second differences when a table of proportional parts is supplied. Thus to find from such a table $\log 232.5648$, from the table

$$\log 232.6 = 2.366610$$

$$\log 232.5 = 2.366423$$

$$\text{1st diff.} = 187 \text{ (in last three places)}$$

$$\text{For next fig. 7 p.p.} = 130.9$$

$$\text{" " " 6 p.p.} = 112.2$$

$$\text{2nd diff.} = 18.7$$

$$\text{For next figs. 48, p.p.} = 8.99 \text{ (by slide rule)}$$

$$\text{For figs. 648, p.p.} = 112.2 + 8.99 = 121.19, \text{ or}$$

$$= 121 \text{ (in last three places)}$$

$$\text{Log } 232.5648 = 2.366544$$

The slide rule setting for the second interpolation is:

C	Set 1	R to 48
D	To 18.7	Read 8.99

Of course, as in the first example given, the logarithm difference 121 could be found just as well by reading directly on the slide rule without reference to the table of proportional parts. Thus by formula (51)

$$\frac{1}{187} = \frac{648}{X} \quad \text{and} \quad X = 121.$$

For a six-place table, where the logarithm differences do not go beyond three figures, therefore, the tables of proportional parts are unnecessary and the slide rule may be used directly for the full difference. With tables of seven and more figures the slide rule may be used as in

the second example above to extend the tables of proportional parts.

For interpolating antilogarithms the procedure given above is simply reversed by taking X as known in proportion (51) and d as unknown. Thus to find antilog 1.452893:

$$\text{antilog } 1.453012 = 28.38$$

$$\text{antilog } 1.452859 = 28.37$$

$$D = 153$$

$$\text{For antilog } 1.452893, X = 34,$$

$$d = .222 \text{ (next figs. after 37)}$$

$$\text{antilog } 1.452893 = 28.37222$$

The slide rule setting for this form of interpolation is:

C	Set 153	R to 1
D	To 34	$d = .222$

Natural logarithms and antilogarithms may be interpolated in the same way as for common logarithms. The natural and logarithmic angular functions may also be interpolated in the same way. Since the procedure for all these is precisely the same as that given above they will not be described here.

CHAPTER IV

TYPICAL PROBLEMS AND SLIDE RULE SETTINGS

A. INTRODUCTION.

The various forms of slide rule using the standard Mannheim scales being adapted to any form of computation involving multiplication, division, proportion, powers, roots, common and natural logarithms, exponentials and trigonometric functions are capable of performing an indefinite number and variety of operations. With any given type of slide rule and problems certain appropriate settings may be developed to suit individual needs. These may be modified, varied or adapted to suit other needs or scale arrangements. As an illustration of the development of such settings there is given below in Section B a list of diagrammed settings for a number of commercial and engineering problems which involve only the A, B, C and D scales. These are of course adapted to any of the standard rules. Lists of settings involving the ordinary and standard operations with these scales, such as those described in Chapter II, and their simple combinations will not be given.

In Section C are given the settings and gauge points

for a number of the more commonly useful geometrical mensuration calculations and for a large number of standard conversions from one to another of the various systems of units of measurement. The gauge points apply to the C and D scales; these settings may therefore be used with any of the standard forms of rules. Many of these conversion gauge points and settings are printed on the back of the stock of the Mannheim rule by most manufacturers.

The settings given in the remaining Sections of this chapter in the main apply particularly to other forms of rule than the simple Mannheim but where they involve only the A, B, C and D scales they may of course also be used on the Mannheim slide rule.

B. TYPICAL PROBLEMS AND SETTINGS FOR THE A, B, C AND D SCALES.

In the thirty-seven typical settings which follow the computing formula is given for many of the problems; many others involve the simple standard operations and the formula is not given with the setting. The method which has been used throughout this book and explained in Chapter II is used. Horizontal lines represent the edges of contiguous scales and the scales are indicated by capital letters above and below these lines in the positions in which they occur on the slide rule. Vertical lines

separate the successive steps in an operation; double vertical lines indicating that the slide is to be reset or removed and reversed or inverted, or serving to separate the letters used to indicate the scale from other letters involved in the setting of the problem. While these settings apply only to the A, B, C and D scales they indicate the method of working out many different types of problem and may serve as models of the methods of combining other scales than these four. Some of these operations may be more conveniently performed by means of other scales, as will appear in later sections devoted to the several types of rules.

1. — Diameters and Areas of Circles. $A = .7854 D^2$

A	To 205		A	To 11	
B	Set 161	Find Areas	B	Set 6	Find Areas in square feet
C			C		
D		Above Diameters	D		Above Diameters in inches

or

2. — To Calculate Selling Prices of Goods, with Percentage of Profit on Cost Price

C	Set 100	Below cost price
D	To 100 plus percentage of profit	Find selling price

3. — To Calculate Selling Prices of Goods, with Percentage of Profit on Selling Price

C	Set 100 less percentage of profit	Below cost price
D	To 100	Find selling price

154 TYPICAL PROBLEMS AND SLIDE RULE SETTINGS

Example: If goods cost 45 cents a yard, at what price must they be sold to realize 15 per cent profit on the selling price?

C		Set 85 (= 100 - 15)		Below 45
D		To 100		Find 53 — Answer

4. — To find the Area of a Ring. $A = \frac{(D + d) \times (D - d)}{1.2732}$

C		Set sum of the two diameters		Find area.
D		To 1.273		Above difference of the two diameters

5. — Compound Interest

Set the left index of C, to 100 plus the rate of interest, on D, then take the corresponding number on the scale of Equal Parts, and multiply it by the number of years. Set this product on the scale of E.P. to the index on the under side of the Rule, then on D will be found the amount of any coinciding sum on C for the given years at the given rate.

Example: Find the amount of \$150 at 5% at the end of 10 years.

C		Set 100		E.P. = 21.2×10 = 212		212 to 1		C		Below \$150
D		To 105		Under side of Rule		and Slide		D		Find \$244.35 — Answer

We thus obtain on D, below 1 on C, a gauge-point for 10 years at 5%, and can obtain in like manner similar ones for any other number of years and rate of interest.

6. — Levers

C		Set distance from fulcrum to power or weight applied		Find power or weight transmitted
D		To distance from fulcrum to power or weight transmitted		Above power or weight applied

7. — Diameter of Pulleys or Teeth of Wheels

in'd C	Set diameter or teeth of DRIVING	Diameter or teeth of DRIVEN
D	To revolutions of DRIVING	Revolutions of DRIVEN
C	Set diameter or teeth of DRIVING	Revolutions of DRIVEN
or, D	To diameter or teeth of DRIVEN	Revolutions of DRIVING

8. — Diameter of Two Wheels to Work at Given Velocities

C	Set distance between their centres	Find diameter
D	To half SUM of their revolutions	Above revolutions of each.

Example: A shaft makes 21 revolutions, and is to drive another shaft which should make 35 revolutions. The distance between their centres is 48 inches. What should be the diameters of the gears?

C	Set 48	Find 36	Find 60
D	To 28(= 21 + 35 ÷ 2)	Above 21	And above 35

The two wheels must therefore be 36 and 60 inches diameter.

9. — Strength of Teeth of Wheels

$$P = \frac{\sqrt{H}}{0.6 V}$$

A	To H.P. to be transmitted	.		
B				
C	Set gauge-point 0.6 R to 1	Velocity in ft. per second to R	Under 1	
D				Pitch in inches.

10. — Diameter and Pitch of Wheels

$$N = \frac{D \times \pi}{P}$$

C	Set 226	R to 1	Pitch to R	Under diameter of pitch circle
D	To 710			Find number of teeth

11. — Strength of Wrought Iron Shafting

$$D = \sqrt[3]{\frac{83 H}{N}} \text{ for crank shafts and prime movers.}$$

$$D = \sqrt[3]{\frac{65 H}{N}} \text{ for ordinary shafting.}$$

A	To 83 or 65		
B	Set revolutions per min.	R to Ind. H.P.	Number coinciding with R = diameter
C			Under 1
D			Same coinciding number = diameter

NOTE. — In this, as in other cases, the coefficients (83 and 65) may be altered to suit individual opinions, without in any way altering the methods of solution.

12. — To find the Change Wheel in a Screw-Cutting Lathe

$$N = T \frac{S \times W}{M \times P} \quad \text{Where} \quad \left\{ \begin{array}{l} N = \text{Number of threads per inch to be cut} \\ T = \text{Number of threads per inch on traverse screw.} \\ M = \text{Number of teeth in wheel on mandril.} \\ W = \text{Number of teeth in stud wheel (gearing in } M \text{).} \\ P = \text{Number of teeth in stud pinion (gearing in } S \text{).} \\ S = \text{Number of teeth in wheel on traverse screw.} \end{array} \right.$$

$$W = N \frac{M \times P}{T \times S}$$

C	Set T	R to P	S to R	Under M
D	To N			Find number of teeth in W or stud wheel.

13. — Rules for Good Leather Belting

$$W = \frac{600 \text{ or } 375 \text{ H.P.}}{V \text{ ft. per min.}}$$

C	Set 600	Find width in inches	for SINGLE BELTS.
D	To velocity in feet per min.	Above actual horse power	

C	Set 375	Find width in inches	for DOUBLE BELTS.
D	To velocity in feet per min.	Above actual horse power	

14. — Best Manilla Rope Driving

A	To velocity in feet per min.	Find ACTUAL HORSE POWER
B	Set 307	Above diameter in inches
C		
D		

A	To 4	Find STRENGTH IN TONS
B		
C	Set 1	Above diameter in inches
D		

A	To 107	Find WORKING TENSION IN POUNDS
B		
C	Set 1	Above diameter in inches
D		

A	To 0.28	Find WEIGHT PER FOOT IN POUNDS
B		
C	Set 1	Above diameter in inches
D		

15. — Weight of Iron Bars in Pounds per Foot Length

A	To 1	Weight of SQUARE BARS
B	Set 3	
C		Above width of side in inches
D		

A	To 55	Weight of ROUND BARS
B	Set 21	
C		Above diameter in inches
D		
C	Set 0.3	Below thickness in inches
D	Breadth in inches	Weight of FLAT BARS

16. — Weight of Iron Plates in Pounds per Square Foot

C	Set 32	Below thickness in thirty-seconds of an inch
D	To 40	Find weight in pounds per square foot

17. — Weights of Other Metals

C	Set 1	Below G.P. for other metals
D	To weight in iron	Find weight in other metals

Gauge-points of other metals, and weight per cubic foot

	W. I.	C. I.	Cast Steel.	Steel Plates.	Copper.	Brass.	Lead.	Cast Zinc.
G. P.	1	.93	1.02	1.04	1.15	1.09	1.47	.92
Weight	480	450	490	500	550	525	710	440 pounds

Example: What is the weight of a bar of copper, 1 foot long, 3 inches broad and 2 inches thick?

C	Set 0.3	R to 2 inches thick	1 to R	Below G.P. 1.15
D	To 3 inches broad			Find 23 pounds — Answer

18. — Weight of Cast Iron Pipes

C	Set .4075	Below DIFFERENCE of inside and outside diameters in inches
D	To SUM of inside and outside diameters in inches	Find weight in pounds per lineal foot

G.P. for other metals	Brass.	Copper.	Lead.	W. Iron.
	.355	.333	.259	.38

19. — Safe Load on Chains

A		Safe load in tons
B	Set 36	Above 1
C		
D	To diameter in sixteenths of an inch	

20. — Gravity

C	Set 1	Below 32.2
D	To seconds	Velocity in feet per second
A	Space fallen through in feet	
B		
C	Set 1	Under 8
D		Velocity in feet per second

A		Space fallen through in feet
B		Above 16.1
C	Set 1	
D	To seconds	

21. — Oscillations of Pendulums

A		
B	Set length pendulum in inches	
C		Below 1
D	To 375	Number oscillations per minute

22. — Comparison of Thermometers

C	Set 5	Degrees CENTIGRADE
D	To 9	Degrees + 32 = FAHRENHEIT

C	Set 4	Degrees REAUMUR
D	To 9	Degrees + 32 = FAHRENHEIT
C	Set 4	Degrees REAUMUR
D	To 5	Degrees CENTIGRADE

23. — Force of Wind

A		
B	Set 10	Find pressure in pounds per square foot
C		
D	To 66	Velocity in FEET <i>per second</i>
A		
B	Set 10	Find pressure in pounds per square foot
C		
D	To 45	Velocity in MILES <i>per hour</i>

24. — Discharge from Pumps

A		Gallons delivered per stroke
B	Set 29.4	Stroke in feet
C		
D	To diameter in inches	

25. — Diameter of Single-acting Pumps

A	To 29.4			
B	Set length stroke in ft.	R to gallons to be delivered per min.	No. strokes per min. to R	
C				Below 1
D				Diam. pump in inches

26. — Horse Power Required for Pumps

C	Set G.P.	Height in feet to which the water is to be raised
D	To cubic feet or gallons to be raised per minute	Horse power required

Gauge Points with different Percentages of Allowance

Per cent	None	10	20	30	40	50	60	70	80
For Gallons, Imp.	3300	3000	2750	2540	2360	2200	2060	1940	1835
For C. Feet	528	480	440	406	377	352	330	311	294
For U. S. Gallons	3960	3600	3300	3050	2830	2640	2470	2330	2200

27. — Theoretic Velocity of Water for any Head

A	Head in feet	
B		
C	Set 1	Under 8
D		Velocity in feet per second

28. — Theoretical Discharge from an Orifice 1 inch Square

A			If the hole is round and 1 inch diameter, the G.P. is 2.62.
B	Set 1	Under head in feet	
C			
D	To G.P. 3.34	Discharge in cubic feet	

29. — Real Discharge from Orifice in a Tank, 1 inch Square

A			If the hole is round and 1 inch diameter, the G.P. is 1.65.
B	Set 1	Under head in feet	
C			
D	To 2.1 G.P.	Discharge in cubic feet per minute with coefficient .63.	

Gauge Points for Other Coefficients

Coefficient . .	.60	.66	.69	.72	.75	.78	.81	.84	.87	.90	.93	.96
G. P. SQUARE, 2.	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.	3.1	3.2	
G. P. ROUND	1.57	1.73	1.80	1.88	1.96	2.04	2.12	2.20	2.28	2.36	2.44	2.52

30. — Discharge from Pipes When Real Velocity Is Known

Inverted {	A		Discharge in cubic feet per minute.
	C		Above 1.75
	B	Velocity in feet per second	
	D	Diameter in inches	

31. — Delivery of Water from Pipes $W = 4.71 \sqrt{\frac{D^5 H}{L}}$

C	Set 1	EP $\times$ 5 = x	x to I	Eytelwein's Rule.
D	To diameter in inches	Under side of Rule		(Con't. to next line.)
A				To Z
B				Set 1
C	R to 1	Length in feet to R	Under head in feet	Under 4.71
D			Find Z	Cubic ft. per min.

This is worked out similarly to Formula 5, which is explained in full.

32. — Gauging Water with a Weir

Inverted {	A		
	C		
	B	Depth in inches	Under 4.3
	D	Depth in inches	Discharge in cubic feet per minute from each foot width of sill.

33. — Discharge of a Turbine

$$\frac{\sqrt{H \times V}}{0.3} = D$$

Inverted {	A		
	C		Under 0.3
	B	Head in feet	
	D	Square inches water vented	Cubic feet discharged per minute.

34. — Revolutions of a Turbine

A	To head in feet			
B				
C	Set diameter in inches	R to 1840	1 to R	Under rate of peripheral velocity
D				Find revolutions per minute.

35. — Horse Power of a Turbine

C	Set 530	R to discharge per c. ft. per min.	1 to R	Percentage useful effect
D	Head in ft.			Horse power

or,	A	Under head in ft.				
	B					
	C	Set 1	R to head in ft.	158 to R	R to vent in sq. in.	1 to R Under useful effect
	D					Horse power.

36. — Horse Power of a Steam Engine

C	Set 21,000	R to diam.	1 to R	R to stroke in ft.	1 to R	R to rev. per min.	1 to R	Mean pressure per sq. inch
D	To diam. in inches							Horse power.

or,	A					Horse power
	B	Set 21,000	R to stroke in ft.	1 to R	R to revolutions	1 to R
	C					Mean pressure
	D	To diam. in inches				

37. — Dynamometer; to Estimate the H.P. Indicated by

C	Set 5252	R to length of lever in feet from center of shaft	1 to R	Under rev. of shaft per min.
D	Weight applied at end of lever in pounds, includ- ing weight of scale			Actual horse power.

C. TABLES OF CONVERSIONS AND GAUGE POINTS FOR C AND D SCALES.

The settings for the following conversions are made in the usual manner, as explained above in Section B. Since all are intended for the same scales, however (C and D), the scales are not indicated in the settings. The numbers and other quantities above the horizontal line in each case are to be read on the C scale and those below the line on the D scale. After the slide is set as indicated by the two numbers given for each setting the runner may be set at the quantity on the C scale or that on D, that is, the setting or conversion may be used in either sense. Thus as an example consider the first setting under GEOMETRICAL below, and let it be required to find the circumference of a circle of 12 inches diameter. With 226 on C set at 710 on D the runner is set at 12 on C and on D is read the circumference 37.7 inches. On the other hand, without moving the slide the same ratio may be used to find the diameter of a circle whose circumference is known to be 56.5 inches; thus set the runner at 56.5 on D and on C read the diameter 18.01 inches. Similarly the first setting under METRIC

SYSTEM may be used to convert inches to centimeters or centimeters to inches, and so for all the others. As further illustration we give here a few examples involving some of the COMBINATIONS.

EXAMPLES

What is the pressure in pounds per square inch equivalent to a head of 34 feet of water?

C		Set 60		Under 34
D		To 26		Find 14.75 pounds — Answer

What head of water, in feet, is equivalent to a pressure of 18 pounds per square inch?

C		Set 26		Under 18
D		To 60		Find 41.5 feet — Answer

How many horse power will 50 cubic feet of water per minute give under a head of 400 feet?

C		Set 3700		Runner to 400		1 to R		Under 50
D		To 7						Find 37.8 H.P. — Answer

In each of the equivalent ratios given in this section the error is less than one-tenth of one percent.

(1) GEOMETRICAL

Set 226 = Diameters of circles.

To 710 = Circumferences of circles.

79 = Diameter of circle.

70 = Side of equal square.

99 = Diameter of circle.

70 = Side of inscribed square.

$$\frac{39 = \text{Circumference of circle.}}{11 = \text{Side of equal square.}}$$

$$\frac{40 = \text{Circumference of circle.}}{9 = \text{Side of inscribed square.}}$$

$$\frac{70 = \text{Side of square.}}{99 = \text{Diagonal of square.}}$$

$$\frac{205 = \text{Area of square whose side} = 1.}{161 = \text{Area of circle whose diameter} = 1.}$$

$$\frac{322 = \text{Area of circle.}}{205 = \text{Area of inscribed square.}}$$

(2) ARITHMETICAL

$$\frac{100 = \text{Links}}{66 = \text{Feet}}$$

$$\frac{12 = \text{Links.}}{95 = \text{Inches.}}$$

$$\frac{101 = \text{Square links.}}{44 = \text{Square feet.}}$$

$$\frac{6 = \text{U. S. gallons.}}{5 = \text{Imperial gallons.}}$$

$$\frac{1 = \text{U. S. gallons.}}{231 = \text{Cubic inches.}}$$

$$\frac{800 = \text{U. S. gallons.}}{107 = \text{Cubic feet.}}$$

$$\frac{22 = \text{Imperial gallons.}}{6100 = \text{Cubic inches.}}$$

$$\frac{430 = \text{Imperial gallons.}}{69 = \text{Cubic feet.}}$$

(3) METRIC SYSTEM

<u>26 = Inches.</u>	<u>82 = Feet.</u>
66 = Centimetres.	25 = Metres.
<u>82 = Yards.</u>	<u>87 = Miles.</u>
75 = Metres.	140 = Kilometres.
<u>4300 = Links.</u>	<u>43 = Chains.</u>
865 = Metres.	865 = Metres.

<u>31 = Square inches.</u>
200 = Square centimetres.
<u>140 = Square feet.</u>
13 = Square metres.
<u>61 = Square yards.</u>
51 = Square metres.
<u>42 = Acres.</u>
17 = Hectares.
<u>22 = Square miles.</u>
57 = Square kilometres.
<u>5 = Cubic inches.</u>
82 = Cubic centimetres.
<u>600 = Cubic feet.</u>
17 = Cubic metres.
<u>85 = Cubic yards.</u>
65 = Cubic metres.
<u>6 = Cubic feet.</u>
170 = Litres.
<u>14 = U. S. gallons.</u>
53 = Litres.
<u>46 = Imperial gallons.</u>
209 = Litres.

$$108 = \text{Grains.}$$

$$7 = \text{Grammes.}$$

$$75 = \text{Pounds.}$$

$$34 = \text{Kilogrammes.}$$

$$6 = \text{Ounces.}$$

$$170 = \text{Grammes.}$$

$$63 = \text{Hundredweights.}$$

$$3200 = \text{Kilogrammes.}$$

$$63 = \text{English Tons.}$$

$$64 = \text{Metric Tonnes.}$$

(4) PRESSURES

$$640 = \text{Pounds per square inch.}$$

$$45 = \text{Kilogs per square centimetre.}$$

$$51 = \text{Pounds per square foot.}$$

$$249 = \text{Kilogs per square metre.}$$

$$59 = \text{Pounds per square yard.}$$

$$32 = \text{Kilogs per square metre.}$$

$$57 = \text{Inches of mercury.}$$

$$28 = \text{Pounds per square inch.}$$

$$82 = \text{Inches of mercury.}$$

$$5800 = \text{Pounds per square foot.}$$

$$720 = \text{Inches of water.}$$

$$26 = \text{Pounds per square inch.}$$

$$74 = \text{Inches of water.}$$

$$385 = \text{Pounds per square foot.}$$

$$60 = \text{Feet of water.}$$

$$26 = \text{Pounds per square inch.}$$

$$5 = \text{Feet of water.}$$

$$312 = \text{Pounds per square foot.}$$

$$15 = \text{Inches of mercury.}$$

$$17 = \text{Feet of water.}$$

99 = Atmospheres.

2960 = Inches of mercury.

34 = Atmospheres.

500 = Pounds per square inch.

34 = Atmospheres.

7200 = Pounds per square foot.

30 = Atmospheres.

31 = Kilogs per square centimetre.

23 = Atmospheres.

780 = Feet of water.

3 = Atmospheres.

31 = Metres of water.

29 = Pounds per square inch.

67 = Feet of water.

1 = Kilogs per square centimetre.

10 = Metres of water.

(5) COMBINATIONS

43 = Pounds per foot.

64 = Kilogs per metre.

127 = Pounds per yard.

63 = Kilogs per metre.

46 = Pounds per square yard.

25 = Kilogs per square metre.

49 = Pounds per cubic foot.

785 = Kilogs per cubic metre.

27 = Pounds per cubic yard.
16 = Kilogs per cubic metre.
89 = Cubic feet per minute.
42 = Litres per second.
700 = Imperial gallons per minute.
53 = Litres per second.
840 = U. S. gallons per minute.
53 = Litres per second.
38 = Weight of fresh water.
39 = Weight of sea water.
5 = Cubic feet of water.
312 = Weight in pounds.
1 = Imperial gallons of water.
10 = Weight in pounds.
3 = U. S. gallons of water.
25 = Weight in pounds.
50 = Pounds per U. S. gallon.
6 = Kilogs per litre.
10 = Pounds per imperial gallon.
1 = Kilogs per litre.
30 = Pounds per U. S. gallon.
25 = Pounds per imperial gallon.
3 = Cubic feet of water.
85 = Weight in kilogs.
46 = Imperial gallons of water.
209 = Weight in kilogs.

14 = U. S. gallons of water.

53 = Weight in kilogs.

44 = Feet per second.

30 = Miles per hour.

88 = Yards per minute.

3 = Miles per hour.

41 = Feet per second.

750 = Metres per minute.¹

82 = Feet per minute.

25 = Metres per minute.

340 = Footpounds.

47 = Kilogrammetres.

72 = British horse power.

73 = French horse power.

3700 = One cubic foot of water per minute under one foot of head.

7 = British horse power.

75 = One litre of water per second under one metre of head.

1 = French horse power.

D. SETTINGS FOR THE POLYPHASE MANNHEIM SLIDE RULE.

In the list below are ninety-four settings which are specially adapted to the Polyphase Mannheim slide rule. (This rule is also manufactured under such names as Maniphase, Multiplex, etc.) The settings are not

diagrammed as in Section B above but are very clearly and concisely stated so that there is no difficulty in performing the operations. These descriptions are merely concise forms of what we have in Chapters II and III called the "statement" of a setting or operation, and although the various steps in each statement are not numbered they are to be carried out in the order given in each case. Although these settings are especially adapted to slide rules carrying the inverted (CI) and cube (K) scales, those involving only the standard scales will of course also apply to the standard Mannheim rule.

Although the present list is fairly long it is not to be understood that it is intended as complete; the settings given are the simpler and more commonly useful ones but there are of course scores of others of which those given are merely illustrative or typical. By a study of those in the list the manner of working out the combination of scales for almost any operation will become evident or be suggested.

In Articles 19 and 26 methods of finding powers and roots higher than the third are given in connection with the A, B and K scales; these methods are used here. Thus the fourth root of a number is obtained by finding the square root of the square root. The sixth root is obtained by finding the square root of the cube root. The eighth root is the square root of the fourth root, etc.

Expressions Which May be Read Directly by Means of the Indicator, without Setting the Slide

1. $x = a^2$, set Indicator to a on D, read x on A.
2. $x = a^3$, set Indicator to a on D, read x on K.
3. $x = \sqrt{a}$, set Indicator to a on A, read x on D.
4. $x = \sqrt[3]{a}$, set Indicator to a on K, read x on D.
5. $x = \sqrt{a^3}$, set Indicator to a on A, read x on K.
6. $x = \sqrt[3]{a^2}$, set Indicator to a on K, read x on A.
7. $x = \frac{1}{a}$, set Indicator to a on CI, read x on C.
8. $x = \frac{1}{a^2}$, set Indicator to a on CI, read x on B.
9. $x = \frac{1}{\sqrt{a}}$, set Indicator to a on B, read x on CI.

With Indices in Alignment

10. $x = \frac{1}{a^3}$, set Indicator to a on CI, read x on K.
11. $x = \frac{1}{\sqrt[3]{a}}$, set Indicator to a on K, read x on CI.

Expressions Solved with One Setting of Slide

SETTINGS FOR ONE FACTOR

12. $x = a^4$, set 1 to a on D, over a on C, read x on A.
13. $x = \frac{1}{a^4}$, set a on CI to a on D, under 1 on A, read x on B.
14. $x = a^5$, set a on CI to a on D, over a on B, read x on A.
15. $x = \frac{1}{a^5}$, set a on C to a on K, under a on CI, read x on K.
16. $x = a^6$, set 1 on C to a on D, under a on C, read x on K.
17. $x = a^7$, set a on CI to a on K, under a on C, read x on K.
18. $x = a^9$, set a on CI to a on D, under a on C, read x on K.
19. $x = \sqrt{a^5}$, set 1 to a on K, under a on B, read x on K.
20. $x = \sqrt{a^9}$, set 1 to a on A, under a on C, read x on K.
21. $x = \sqrt{a^{11}}$, set a on CI to a on K, under a on B, read x on K.
22. $x = \sqrt{a^{15}}$, set a on CI to a on D, under a on B, read x on K.

23. $x = \frac{1}{\sqrt[3]{a^2}}$, set 1 to a on K, under 1 on A, read x on B.
24. $x = \sqrt[3]{a^4}$, set 1 to a on K, under a on C, read x on D.
25. $x = \frac{1}{\sqrt[3]{a^4}}$, set a on CI to a on K, over 1 on D, read x on C.
26. $x = \sqrt[3]{a^5}$, set 1 to a on K, over a on B, read x on A.
27. $x = \frac{1}{\sqrt[3]{a^5}}$, set a on C to a on K, under a on CI, read x on D.
28. $x = \sqrt[3]{a^7}$, set a on CI to a on K, under a on C, read x on D.
29. $x = \sqrt[3]{a^8}$, set 1 to a on K, over a on C, read x on A.
30. $x = \frac{1}{\sqrt[3]{a^8}}$, set a on CI to a on K, under 1 on A, read x on B.
31. $x = \frac{1}{\sqrt[3]{a^{10}}}$, set a on C to a on K, over a on CI, read x on A.
32. $x = \sqrt[3]{a^{11}}$, set a on CI to a on K, over a on B, read x on A.
33. $x = \sqrt[3]{a^{14}}$, set a on CI to a on K, over a on C, read x on A.
34. $x = \sqrt[6]{a}$, set a on B to a on K, over 1 on D, read x on C.
35. $x = \frac{1}{\sqrt[6]{a}}$, set a on B to a on K, under 1 on C, read x on D.
36. $x = \sqrt[6]{a^5}$, set 1 to a on K, under a on B, read x on D.
37. $x = \sqrt[6]{a^7}$, set a on B to a on K, over a on D, read x on C.
38. $x = \frac{1}{\sqrt[6]{a^7}}$, set a on B to a on K, over a on D, read x on CI.
39. $x = \sqrt[6]{a^{11}}$, set a on CI to a on K, under a on B, read x on D.

SETTINGS FOR TWO FACTORS

40. $x = ab$, set 1 to a on D, under b on C, read x on D.
41. $x = \frac{1}{ab}$, set a on CI to b on D, over 1 on D, read x on C.
42. $x = \frac{a}{b}$, set b on C to a on D, under 1 on C, read x on D.
43. $x = \frac{1}{a'b}$, set b on C to a on D, over 1 on D, read x on C.

44. $x = ab^2$, set 1 to a on A, over b on C, read x on A.
45. $x = \frac{1}{ab^2}$, set b on CI to a on A, under 1 on A, read x on B.
46. $x = \frac{a}{b^2}$, set b on C to a on A, over 1 on B, read x on A.
47. $x = \frac{a^2}{b}$, set a on C to b on A, under 1 on A, read x on B.
48. $x = a^2b^2$, set 1 on C to a on D, over b on C, read x on A.
49. $x = \frac{1}{a^2b^2}$, set a on CI to b on D, under 1 on A, read x on B.
50. $x = \frac{1}{a^3b^3}$, set a on C to 1 on D, under b on CI, read x on K.
51. $x = \frac{a^3}{b^2}$, set b on C to a on D, at 1 on C, read x on A.
52. $x = a\sqrt{b}$, set 1 to a on D, under b on B, read x on D.
53. $x = \frac{1}{a\sqrt{b}}$, set a on CI to b on A, over 1 on D, read x on C.
54. $x = \frac{\sqrt{a}}{b}$, set b on C to a on A, under 1 on C, read x on D.
55. $x = \frac{a}{\sqrt{b}}$, set b on B to a on D, under 1 on C, read x on D.
56. $x = a^2\sqrt{b}$, set a on CI to a on D, under b on B, read x on D.
57. $x = a^4b$, set a on CI to b on A, over a on C, read x on A.
58. $x = a^6\sqrt{b^3}$, set a on CI to b on A, under a on C, read x on K.
59. $x = \frac{a^2}{\sqrt{b}}$, set b on B to a on D, under a on C, read x on D.
60. $x = \frac{\sqrt{a}}{b^2}$, set b on C to a on A, under b on CI, read x on D.
61. $x = ab^3$, set 1 on C to a on K, under b on C, read x on K.
62. $x = \frac{a}{b^3}$, set b on C to a on K, under 1 on C, read x on K.
63. $x = \frac{a^3}{b^2}$, set a on CI to a on A, over b on CI, read x on A.
64. $x = \frac{a^2}{b^3}$, set b on B to a on D, over b on CI, read x on A.
65. $x = a^2b^2$, set b on CI to a on D, over b on B, read x on A.
66. $x = a\sqrt[3]{b}$, set 1 on C to b on K, under a on C, read x on D.

67. $1 \div a\sqrt[3]{b}$, set a on CI to b on K, over 1 on D, read x on C.
 68. $x = a \div \sqrt[3]{b}$, set a on C to b on K, over 1 on D, read x on C.
 69. $x = \sqrt[3]{a} \div b$, set b on C to a on K, under 1 on C, read x on D.
 70. $x = a^3b^3$, set 1 on C to b on D, under a on C, read x on K.
 71. $x = \frac{a^3}{b^3}$, set b on C to a on D, under 1 on C read x on K.
 72. $x = ab^4$, set b on CI to a on A, over b on C, read x on A.
 73. $x = b^6\sqrt{a^3}$, set b on CI to a on A, under b on C, read x on K.
 74. $x = a^2\sqrt[3]{b^2}$, set 1 on C to b on K, over a on C, read x on A.
 75. $x = \frac{\sqrt[3]{a^4}}{b}$, set b on C to a on K, under a on C, read x on D.
 76. $x = \frac{\sqrt[3]{a^8}}{b^2}$, set b on C to a on K, over a on C, read x on A.
 77. $x = \frac{a^4}{b^3}$, set b on C to a on K, over a on C, read x on K.

SETTINGS FOR THREE FACTORS

78. $x = a \cdot b \cdot c$, set a on CI to b on D, under c on C, read x on D.
 79. $x = a^2 \times b^2 \times c^2$, set a on CI to b on D, over c on C, read x on A.
 80. $x = a^3 \times b^3 \times c^3$, set a on CI to b on D, over c on C, read x on K.
 81. $x = \frac{a \times b}{c}$, set c on C to a on D, under b on C, read x on D.
 82. $x = \frac{a^2b^2}{c^2}$, set c on C to a on D, over b on C, read x on A.
 83. $x = \frac{a^3b^3}{c^3}$, set c on C to a on D, over b on C, read x on K.
 84. $x = \frac{a}{b \cdot c}$, set b on C to a on D, under c on CI, read x on D.
 85. $x = \frac{a^2}{b^2 \times c^2}$, set b on C to a on D, over c on CI, read x on A.
 86. $x = \frac{a^3}{b^3 \times c^3}$, set b on C to a on D, over c on CI, read x on K.
 87. $x = ab\sqrt{c}$, set a on CI to c on A, under b on C, read x on D.
 88. $x = a^2b^2c$, set a on CI to c on A, over b on C, read x on A.
 89. $x = a^3b^3\sqrt{c^3}$, set a on CI to c on A, over b on C, read x on K.
 90. $x = ab\sqrt[3]{c}$, set a on CI to c on K, under b on C, read x on D.

91. $x = a^2 b^2 \sqrt[3]{c^2}$, set a on CI to c on K, over b on C, read x on A.
92. $x = a^3 \cdot b^3 \cdot c$, set a on CI to c on K, over b on C, read x on K.
93. $x = \frac{\sqrt{a}}{b \sqrt[3]{c}}$, set b on CI to c on K, under a on A, read x on C.
94. $x = \frac{a \sqrt{b}}{\sqrt[3]{c}}$, set a on C, to c on K, under b on A, read x on C.

And scores of other combinations.

E. SETTINGS FOR THE POLYPHASE DUPLEX AND LOG-LOG DUPLEX SLIDE RULES.

In the following list of 153 settings the method of writing the formulas and stating the settings is the same as that used above in Section D. Since the Log-Log Duplex rule carries all the scales found on the Polyphase Duplex these settings are equally well adapted to either rule. It is to be understood in using each of these rules that when a setting of slide or runner (indicator) or both has been made it may be necessary to turn the rule over in order to read the result which may be on the face opposite from the one on which the first steps in the setting were carried out; also it may be necessary to turn the rule over several times in the course of a complete operation of several steps. It is also to be understood as in Section D that this list is by no means exhaustive.

Expressions Read Directly by Means of the Indicator without Setting the Slide

1. $x = a^2$, set Indicator to a on D, read x on A.
2. $x = a^3$, set Indicator to a on D, read x on K.
3. $x = \sqrt{a}$, set Indicator to a on A, read x on D.

4. $x = \sqrt[3]{a}$, set Indicator to a on K, read x on D.
5. $x = \sqrt{a^3}$, set Indicator to a on A, read x on K.
6. $x = \sqrt[3]{a^2}$, set Indicator to a on K, read x on A.
7. $x = \frac{1}{a}$, set Indicator to a on CI, read x on C.
8. $x = a \times \pi$, set Indicator to a on D, read x on DF.
9. $x = \frac{a}{\pi}$, set Indicator to a on DF, read x on D.
10. $x = \frac{\pi}{a}$, set Indicator to a on CF, read x on CI.
11. $x = \frac{1}{a \times \pi}$, set Indicator to a on C, read x on CIF.
12. $x = \pi \sqrt{a}$, set Indicator to a on A, read x on DF.
13. $x = \pi \sqrt[3]{a}$, set Indicator to a on K, read x on DF.
14. $x = \frac{a^2}{\pi^2}$, set Indicator to a on DF, read x on A.
15. $x = \frac{a^3}{\pi^3}$, set Indicator to a on DF, read x on K.
16. $x = \pi \sqrt{\sin a}$, set Indicator to a on S, read x on CF.
17. $x = \frac{1}{\pi \sqrt{\sin a}}$, set Indicator to a on S, read x on CIF.
18. $x = \pi \tan a$, set Indicator to a on T, read x on CF.
19. $x = \frac{1}{\pi \tan a} = \frac{\cot a}{\pi}$, set Indicator to a on T, read x on CIF.
20. $x = \frac{1}{\tan a} = \cot a$, set Indicator to a on T, read x on CI.
21. $x = \log a$, set Indicator to a on D, read x on L.
22. $x = \log \sqrt{a}$, set Indicator to a on A, read x on L.
23. $x = \log \sqrt[3]{a}$, set Indicator to a on K, read x on L.
24. $x = \log \frac{a}{\pi}$, set Indicator to a on DF, read x on L.
25. $x = \frac{1}{a^2}$, set Indicator to a on CI, read x on B.
26. $x = \frac{1}{\sqrt{a}}$, set Indicator to a on B, read x on CI.

With indices in alignment

27. $x = \tan a$, set Indicator to a on T, read x on D.
28. $x = \frac{1}{a^3}$, set Indicator to a on CI, read x on K.
29. $x = \frac{1}{\pi^2 a^2}$, set Indicator to a on CIF, read x on A.
30. $x = \frac{1}{\pi^3 a^3}$, set Indicator to a on CIF, read x on K.
31. $x = \frac{1}{\sqrt[3]{a}}$, set Indicator to a on K, read x on CI.
32. $x = \frac{1}{\pi \sqrt{a}}$, set Indicator to a on A, read x on CIF.
33. $x = \frac{1}{\pi \sqrt[3]{a}}$, set Indicator to a on K, read x on CIF.

Expressions Solved at One Setting of Slide

SETTINGS FOR ONE FACTOR

34. $x = a^4$, set 1 to a on D, at a on C, read x on A.
35. $x = \frac{1}{a^4}$, set a on CI to a on D, under 1 on A, read x on B.
36. $x = a^5$, set a on CI to a on D, over a on B, read x on A.
37. $x = \frac{1}{a^5}$, set a on C to a on K, at a on CI, read x on K.
38. $x = a^6$, set 1 on C to a on D, at a on C, read x on K.
39. $x = a^7$, set a on CI to a on K, at a on C, read x on K.
40. $x = a^8$, set a on CI to a on D, at a on C, read x on K.
41. $x = \sqrt{a^5}$, set 1 to a on K, at a on B, read x on K.
42. $x = \sqrt{a^9}$, set 1 to a on A, at a on C, read x on K.
43. $x = \sqrt{a^{11}}$, set a on CI to a on K, at a on B, read x on K.
44. $x = \sqrt{a^{15}}$, set a on CI to a on D, at a on B, read x on K.
45. $x = \frac{1}{\sqrt[3]{a^2}}$, set 1 to a on K, at 1 on A, read x on B.
46. $x = \sqrt[3]{a^4}$, set 1 to a on K, at a on C, read x on D.
47. $x = \frac{1}{\sqrt[3]{a^4}}$, set a on CI to a on K, at 1 on D, read x on C.

48. $x = \sqrt[3]{a^5}$, set 1 to a on K, at a on B, read x on A.
49. $x = \frac{1}{\sqrt[3]{a^5}}$, set a on C to a on K, at a on CI, read x on D.
50. $x = \sqrt[3]{a^7}$, set a on CI to a on K, at a on C, read x on D.
51. $x = \sqrt[3]{a^8}$, set 1 to a on K, at a on C, read x on A.
52. $x = \frac{1}{\sqrt[3]{a^8}}$, set a on CI to a on K, at 1 on A, read x on B.
53. $x = \frac{1}{\sqrt[3]{a^{10}}}$, set a on C to a on K, at a on CI, read x on A.
54. $x = \sqrt[3]{a^{11}}$, set a on CI to a on K, over a on C, read x on A.
55. $x = \sqrt[3]{a^{14}}$, set a on CI to a on K, at a on B, read x on A.
56. $x = \sqrt[6]{a}$, set a on B to a on K, at 1 on D, read x on C.
57. $x = \frac{1}{\sqrt[6]{a}}$, set a on B to a on K, at 1 on C, read x on D.
58. $x = \sqrt[6]{a^5}$, set 1 to a on K, at a on B, read x on D.
59. $x = \sqrt[6]{a^7}$, set a on B to a on K, at a on D, read x on C.
60. $x = \frac{1}{\sqrt[6]{a^7}}$, set a on B to a on K, at a on D, read x on CI.
61. $x = \sqrt[6]{a^{11}}$, set a on CI to a on K, at a on B, read x on D.

SETTINGS FOR TWO FACTORS

62. $x = ab$, set 1 to a on D, under b on C, read x on D.
63. $x = \frac{1}{ab}$, set a on CI to b on D, over 1 on D, read x on C.
64. $x = \frac{a}{b}$, set b on C to a on D, under 1 on C, read x on D.
65. $x = \frac{1}{\frac{a}{b}}$, set b on C to a on D, over 1 on D, read x on C.
66. $x = ab^2$, set 1 to a on A, over b on C, read x on A.
67. $x = \frac{1}{ab^2}$, set b on CI to a on A, under 1 on A, read x on B.
68. $x = \frac{a}{b^2}$, set b on C to a on A, over 1 on B, read x on A.

69. $x = \frac{a^{\frac{1}{b}}}{b}$, set a on C to b on A, under 1 on A, read x on B.
70. $x = a^2b^2$, set 1 on C to a on D, over b on C, read x on A.
71. $x = \frac{a^{\frac{1}{b^2}}}{a^2b^2}$, set a on C to 1 on D, over b on CI, read x on A.
72. $x = \frac{1}{a^3b^3}$, set a on C to 1 on D, over b on CI, read x on K.
73. $x = \frac{a^2}{b^2}$, set b on C to a on D, over 1 on C, read x on A.
74. $x = a\sqrt{a}$, set 1 on C to b on A, under a on C, read x on D.
75. $x = \frac{1}{a\sqrt{b}}$, set a on CI to b on A, over D, read x on C.
76. $x = \frac{\sqrt{a}}{b}$, set b on C to a on A, under 1 on C, read x on D.
77. $x = \frac{a}{\sqrt{b}}$, set b on B to a on D, under 1 on CI, read x on D.
78. $x = a^2\sqrt{b}$, set a on CI to a on D, under b on B, read x on D.
79. $x = a^4b$, set a on CI to b on A, over a on C, read x on A.
80. $x = a^6\sqrt{b^3}$, set a on CI to b on A, over 1 on C, read x on K.
81. $x = \frac{a^{\frac{1}{b}}}{\sqrt{b}}$, set b on B to a on D, over a on C, read x on D.
82. $x = \frac{\sqrt{a}}{b^2}$, set b on C to a on A, under b on CI, read x on D.
83. $x = ab^3$, set 1 on C to a on K, at b on C, read x on K.
84. $x = \frac{a}{b^3}$, set b on C to a on K, at 1 on C, read x on K.
85. $x = \frac{a^{\frac{1}{b^2}}}{b^2}$, set a on CI to a on A, over b on CI, read x on A.
86. $x = \frac{a^2}{b^3}$, set b on B to a on D, over b on CI, read x on A.
87. $x = a^2b^3$, set b on CI to a on D, over b on B, read x on A.
88. $x = \frac{a}{\sqrt[3]{b}}$, set 1 on C to b on K, over a on D, read x on A.
89. $x = \frac{1}{a\sqrt[3]{b}}$, set a on CI to b on K, over 1 on D, read x on C.
90. $x = \frac{a}{\sqrt[3]{b}}$, set a on C to b on K, over 1 on D, read x on C.

91. $x = \frac{\sqrt[3]{a}}{b}$, set b on C to a on K, under 1 on C, read x on D.
 92. $x = a^2b^3$, set 1 on C to b on D, at a on C, read x on K.
 93. $x = \frac{a^3}{b^3}$, set b on C to a on D, at 1 on C, read x on K.
 94. $x = ab^4$, set b on CI to a on A, over b on C, read x on A.
 95. $x = \sqrt[6]{a^3}$, set b on CI to a on A, at b on C, read x on K.
 96. $x = a^2\sqrt[3]{b^2}$, set 1 on C to b on K, under a on C, read x on A.
 97. $x = \frac{\sqrt[3]{a^4}}{b}$, set b on C to a on K, under a on C, read x on D.
 98. $x = \frac{\sqrt[3]{a^8}}{b^2}$, set b on C to a on K, over a on C, read x on A.
 99. $x = \frac{a^4}{b^3}$, set b on C to a on K, over a on C, read x on K.

SETTINGS FOR TWO FACTORS AND π

100. $x = a \cdot b \cdot \pi$, set 1 on C to a on D, over b on C, read x on DF.
 101. $x = \frac{\pi}{ab}$, set a on CI to b on D, over 1 on D, read x on CF.
 102. $x = \frac{ab}{\pi}$, set a on CI to b on D, over 1 on D, read x on CIF.
 103. $x = \frac{\pi^2a}{b^2}$, set b on CF to a on A, over 1 on C, read x on A.
 104. $x = \frac{\pi\sqrt{a}}{b}$, set b on CF to a on A, under 1 on C, read x on D.
 105. $x = \frac{\pi^2}{a^2b^2}$, set a on CF to 1 on D, over b on CI, read x on A.
 106. $x = \frac{\pi^3}{a^3b^3}$, set a on CF to 1 on D, over b on CI, read x on K.
 107. $x = \frac{\pi^2a^2}{b^2}$, set b on CF to 1 on D, over a on C, read x on A.
 108. $x = \frac{\pi^3a^3}{b^3}$, set b on CF to 1 on D, over a on C, read x on K.
 109. $x = \frac{a}{\pi b}$, set 1 on C to a on D, under b on CIF, read x on D.
 110. $x = \frac{a^2}{\pi^2b^2}$, set 1 on C to a on D, over b on CIF, read x on K.

111. $x = \frac{a^3}{\pi^3 b_c}$, set 1 on C to a on D, over b on CIF, read x on K.
112. $x = \pi a^2 \sqrt{b}$, set a on CI to b on A, over a on C, read x on DF.
113. $x = \frac{\pi a}{\sqrt{b}}$, set a on C to b on A, over 1 on D, read x on CF.
114. $x = \frac{\sqrt{b}}{\pi a}$, set a on C to b on A, over 1 on D, read x on CIF.
115. $x = \frac{\pi a^2}{\sqrt{b}}$, set a on C to b on A, over a on D, read x on CF.
116. $x = \frac{\sqrt{b}}{\pi a^2}$, set a on C to b on A, over a on D, read x on CIF.
117. $x = \frac{\pi \sqrt[3]{a}}{b}$, set b on C to a on K, over 1 on C, read x on DF.
118. $x = \pi \sqrt{a^5}$, set a on CI to a on A, over a on C, read x on DF.
119. $x = \pi \sqrt[3]{a^4}$, set 1 to a on K, over a on C, read x on DF.
120. $x = \frac{\pi}{\sqrt[3]{a^4}}$, set a on CI to a on K, over 1 on D, read x on CF.
121. $x = \frac{\pi}{\sqrt[3]{a^5}}$, set a on C to a on K, over a on CI, read x on DF.
122. $x = \pi \sqrt[3]{a^7}$, set a on CI to a on K, over a on C, read x on DF.
123. $x = \frac{\pi}{\sqrt[3]{a^2}}$, set a on C to a on K, at 1 on C, read x on DF.
124. $x = \frac{\pi}{\sqrt[3]{a^5}}$, set a on C to a on K, at a on CI, read x on DF.
125. $x = \frac{\pi \sqrt[3]{a^4}}{b}$, set b on C to a on K, over a on C, read x on DF.
126. $x = \frac{\pi}{a \sqrt[3]{b}}$, set a on CI to b on K, over 1 on D, read x on CF.
127. $x = \frac{a \sqrt[3]{b}}{\pi}$, set a on CI to b on K, over 1 on D, read x on CIF.

SETTINGS FOR THREE FACTORS

128. $x = a \cdot b \cdot c$, set a on CI to b on D, under c on C, read x on D.
129. $x = a^2 \times b^2 \times c^2$, set a on CI to b on D, over c on C, read x on A.
130. $x = a^3 \times b^3 \times c^3$, set a on CI to b on D, over c on C, read x on K.

131. $x = \frac{a \times b}{c}$, set c on C to a on D, under b on C, read x on D.
132. $x = \frac{a^2 b^2}{c^2}$, set c on C to a on D, over b on C, read x on A.
133. $x = \frac{a^3 b^3}{c^3}$, set c on C to a on D, over b on C, read x on K.
134. $x = \frac{a}{b \cdot c}$, set b on C to a on D, under c on CI, read x on D.
135. $x = \frac{a^2}{b^2 \times c^2}$, set b on C to a on D, over c on CI, read x on A.
136. $x = \frac{a^2}{b^3 \times c^3}$, set b on C to a on D, over c on CI, read x on K.
137. $x = ab\sqrt{c}$, set a on CI to c on A, under b on C, read x on D.
138. $x = a^2 b^2 c$, set a on CI to c on A, over b on C, read x on A.
139. $x = a^3 b^3 \sqrt{c^3}$, set a on CI to c on A, over b on C, read x on K.
140. $x = ab\sqrt[3]{c}$, set a on CI to c on K, under b on C, read x on D.
141. $x = a^2 b^2 \sqrt[3]{c^2}$, set a on CI to c on K, over b on C, read x on A.
142. $x = a^3 \cdot b^3 \cdot c$, set a on CI to c on K, over b on C, read x on K.
143. $x = \frac{\sqrt{a}}{b\sqrt[3]{c}}$, set b on CI to c on K, under a on A, read x on C.
144. $x = \frac{a\sqrt{b}}{\sqrt[3]{c}}$, set a on C, to c on K, under b on A, read x on C.

And scores of other combinations.

SETTINGS FOR THREE FACTORS AND π

145. $a \cdot b \cdot c \cdot \pi$, set a on CI to b on D, over c on C, read x on DF.
146. $x = \frac{ab\pi}{c}$, set c on C to a on D, over b on C, read x on DF.
147. $x = \frac{a\pi}{bc}$, set b on C to a on D, over c on CI, read x on DF.
148. $x = ab\pi\sqrt{c}$, set a on CI to c on A, over b on C, read x on DF.
149. $x = ab\pi\sqrt[3]{c}$, set a on CI to c on K, over b on C, read x on DF.
150. $x = \frac{\pi\sqrt{a}}{b\sqrt[3]{c}}$, set b on CI to c on K, under a on A, read x on CF.
151. $x = \frac{b\sqrt[3]{c}}{\pi\sqrt{a}}$, set b on CI to c on K, under a on A, read x on CIF.

152. $x = \frac{\pi a \sqrt{b}}{\sqrt[3]{c}}$, set a on C to c on K, under b on A, read $\tilde{x}$ on CF.

153. $x = \frac{\sqrt[3]{c}}{\pi a \sqrt{b}}$, set a on C to c on K, under b on A, read x on CIF.

F. SETTINGS AND TYPICAL PROBLEMS INVOLVING THE LOG-LOG SCALES.

In the last list above (Section E) are given settings which involve all the scales of the Log-log Duplex slide rule except the log-log scales. In the following list, therefore, are included only a few such settings, the greater part of the list consisting of settings which involve the log-log scales.

Following the list of sixty-one settings and numbered continuously with these are given the diagrams for twenty-eight typical formulas and problems from various branches of physics and engineering which involve higher powers and roots and exponentials and for which the log-log scales are particularly adapted. By studying these problems and their settings one can easily work out the settings for similar types of problems. For purposes of comparison with the types of settings given in Section B, the first three problems in this second part of the list are taken from Section B and are worked out both with and without the use of the log-log scales or by combination of the two methods.

Expressions which may be Read Directly by Means of the
Indicator, without Setting the Slide

1. $x = a^2$, set Indicator to a on D, read x on A.
2. $x = a^3$, set Indicator to a on D, read x on K.
3. $x = a^{10}$, set Indicator to a on LL1, read x on LL2; or set Indicator to a on LL2, read x on LL3.
4. $x = a^{100}$, set Indicator to a on LL1, read x on LL3.
5. $x = \sqrt{a}$, set Indicator to a on A, read x on D.
6. $x = \sqrt[10]{a}$, set Indicator to a on LL3, read x on LL2; or set Indicator to a on LL2, read x on LL1.
7. $x = \sqrt[100]{a}$, set Indicator to a on LL3, read x on LL1.
8. $x = \sqrt[3]{a}$, set Indicator to a on K, read x on D.
9. $x = \sqrt{a^3}$, set Indicator to a on A, read x on K.
10. $x = \sqrt[3]{a^2}$, set Indicator to a on K, read x on A.
11. $x = \frac{1}{a}$, set Indicator to a on CI, read x on C.
12. $x = a \times \pi$, set Indicator to a on D, read x on DF.
13. $x = \frac{a}{\pi}$, set Indicator to a on DF, read x on D.
14. $x = \frac{\pi}{a}$, set Indicator to a on CF, read x on CI.
15. $x = \frac{1}{a \times \pi}$, set Indicator to a on C, read x on CIF.
16. $x = \pi \sqrt{a}$, set Indicator to a on A, read x on DF.
17. $x = \pi \sqrt[3]{a}$, set Indicator to a on K, read x on DF.
18. $x = \frac{a^{\frac{1}{2}}}{\pi^2}$, set Indicator to a on DF, read x on A.
19. $x = \frac{a^{\frac{1}{3}}}{\pi^3}$, set Indicator to a on DF, read x on K.
20. $x = \pi \sqrt{\sin a}$, set Indicator to a on S, read x on CF.
21. $x = \frac{1}{\pi \sqrt{\sin a}}$, set Indicator to a on S, read x on CIF.
22. $x = \pi \tan a$, set Indicator to a on T, read x on CF.
23. $x = \frac{1}{\pi \tan a} = \frac{\cot a}{\pi}$, set Indicator to a on T, read x on CIF.
24. $x = \tan a$, set Indicator to a on T, read x on C.

25. $x = \frac{1}{\tan a} = \cot a$, set Indicator to a on T, read x on CI.
26. $x = \log a$, set Indicator to a on D, read x on L.
27. $x = \log_e a$, set Indicator to a on LL1, LL2 or LL3, read x on D.
28. $x = \operatorname{colog}_e a$, set Indicator to a on LL0, read x on A.
29. $x = \log \sqrt{a}$, set Indicator to a on A, read x on L.
30. $x = \log \sqrt[3]{a}$, set Indicator to a on K, read x on L.
31. $x = \log \frac{a}{x}$, set Indicator to a on DF, read x on L.

With Indices in Alignment

32. $x = \frac{1}{a^2}$, set Indicator to a on CI, read x on A.
33. $x = \frac{1}{a^3}$, set Indicator to a on CI, read x on K.
34. $x = \frac{1}{\pi^2 a^2}$, set Indicator to a on CIF, read x on A.
35. $x = \frac{1}{\pi^3 a^3}$, set Indicator to a on CIF, read x on K.
36. $x = \frac{1}{\sqrt{a}}$, set Indicator to a on A, read x on CI.
37. $x = \frac{1}{\sqrt[3]{a}}$, set Indicator to a on K, read x on CI.
38. $x = \frac{1}{\pi \sqrt{a}}$, set Indicator to a on A, read x on CIF
39. $x = \frac{1}{\pi \sqrt[3]{a}}$, set Indicator to a on K, read x on CIF.

Expressions Solved at One Setting of Slide

40. $x = a^4$, set 1 to a on D, at a on C, read x on A; or set 1 to a on LL, at 4 on B or C, read x on LL.
41. $x = \frac{1}{a^5}$, set a on C to a on K, at a on CI, read x on K.
42. $x = a^6$, set 1 on C to a on D, at a on C, read x on K; or set 1 to a on LL, at 6 on B or C, read x on LL.

43. $x = a^7$, set a on CI to a on K, at a on C, read x on K; or set 1 to a on LL, at 7 on B or C, read x on LL.
44. $x = a^9$, set a on CI to a on D, at a on C, read x on K; or set 1 to a on LL, at 9 on B or C, read x on LL.
45. $x = a^n$, set 1 to a on LL, at n on B or C, read x on LL.
46. $x = \sqrt[n]{a}$, set n to a on LL, at 1 on B or C, read x on LL.
47. $x = \sqrt{a^3}$, set 1 to a on A, at a on C, read x on K.
48. $x = \sqrt[3]{a^4}$, set 1 to a on K, at a on C, read x on D.
49. $x = \frac{1}{\sqrt[3]{a^4}}$, set a on CI, to a on K, at 1 on D, read x on C.
50. $x = \frac{1}{\sqrt[3]{a^5}}$, set a on C to a on K, at a on CI, read x on D.
51. $x = \sqrt[3]{a^7}$, set a on CI to a on K, at a on C, read x on D.
52. $x = \sqrt[3]{a^8}$, set 1 to a on K, at a on C, read x on A.
53. $x = \frac{1}{\sqrt[3]{a^{10}}}$, set a on C to a on K, at a on CI, read x on A.
54. $x = \sqrt[3]{a^{14}}$, set a on CI to a on K, over a on C, read x on A.
55. $x = \frac{1}{ab}$, set a on CI to b on D, at 1 on D, read x on C.
56. $x = a^5$, set a on CI to a on A, over a on C, read x on A.
57. $x = \sqrt{a^5}$, set a on CI to a on A, at a on C, read x on D.
58. $x = \frac{1}{\sqrt[3]{a^2}}$, set a on C to a on K, at 1 on C, read x on D.
59. $x = \frac{1}{\sqrt[3]{a^5}}$, set a on C to a on K, at a on CI, read x on D.
60. $x = \log a$, set 1 on C to 10 on LL3, at a on LL1, LL2, or LL3, read x on C.
61. $x = \text{colog } a$, set 1 to 10 on LL0, at a on LL0, read x on B.
62. Strength of Wrought Iron Shafting:

$$D = \sqrt[3]{\frac{83 H}{N}} \text{ for crank shafts and prime movers}^1$$

$$D = \sqrt[3]{\frac{65 H}{N}} \text{ for ordinary shafting}$$

C	Set R.P.M.	Indicator to I.H.P.	
D	To 83 or 65	Read diameter ³	Read dia.
K			Opposite dia. ³
or			
C	Set 3		Opposite index
D			
LL	To diameter ³	Read diameter	

NOTE. — In this, as in other cases, the coefficients (83 and 65) may be altered to suit individual opinions, without in any way altering the methods of solution.

63. Compound Interest:

When r is the interest rate expressed in hundredths of a dollar, n is the number of interest periods (years, half-years, quarters, etc.), P is the principal in dollars, and A is the amount in dollars, the formula is

$$A = P(1 + r)^n$$

therefore $\text{Log } A = \text{Log } P + n \text{Log}(1 + r)$

Opposite 1 plus the rate of interest on D, find the corresponding number on the L scale and multiply it by the number of years.

Set the indicator to this product on the L scale.

Set index of C to the indicator.

Opposite the principal on C, read the amount for the given number of years at the given rate on D

D	Set Ind. to 105			Read \$244.35 Ans.
L	Read .0212	Ind. to $.2(.0212 \times 10)$		
C			Left index to ind.	Opposite 150

This problem is solved simply on the *Log-Log scales* as follows:

$$A = P(1 + r)^n$$

$$1 + r = 1.05$$

$$n = 10$$

To 1.05 on LL1, set indicator.

Since the values on the LL2 scale are the 10th powers of the quantities on the LL1 scale, read 1.63 under the indicator on the LL2 scale.

$$1.63 = (1 + r)^n$$

To 1.63 on D, set left index of C.

Below 150 (P) on C, read 244.35 on D.

We thus obtain on D, below 1 on C, a gauge-point for 10 years at 5 per cent and can obtain in like manner similar ones for any other number of years and rate of interest.

64. Delivery of Water from Pipes:

$$W = 4.71 \sqrt{\frac{D^5 H}{L}}$$

Eytelwein's Rule

L	Read log of dia.	Log dia $\times 5 = x$		
D	Opp. dia. in inch.			Read cu. ft. per min.
A		Opposite x		
B		Set length in feet	Opp. head in feet	
C			Set index	Opposite 4.71

When setting x on A do not include characteristic.

Or: —

LL	To dia. in in.	Read dia. ^{.5}			
C	Set index	Below 5		Index to indicator	Opposite 4.71
A			Opposite dia. ^{.5}		
B			Set length in feet	Indicator to head in feet	
D					Read cubic feet per minute

65. Cylindrical Columns:

For solid cylindrical columns of cast iron, both ends rounded, the length of the column exceeding 15 times the diameter

$$P = 33,380 \frac{d^{3.76}}{L^{1.7}}$$

where P = crushing weight in pounds; d = exterior diameter in inches; L = length in feet.

LL	To d	Read $d^{3.76}$	To L	Read $L^{1.7}$		
C	Set 1	Below 3.76	Set 1	Below 1.7	Set $L^{1.7}$	Below 33,380
D					To $d^{3.76}$	Read P

66. Resistance of Hollow Cylinders to Collapse:

Fairbairn's Empirical Formula

$$p = 9,675,000 \frac{t^{2.19}}{ld}$$

where p = pressure in lbs. per square inch; t = thickness of cylinder in inches; d = diameter in inches; l = length in inches.

LL	To t	Read $t^{6.15}$		
B or C	Set 1	Opposite 2.19		
D			To $t^{6.15}$	At Indicator read p
C			Set d	
CI			Indicator to l	Set 9,675,000 to Ind.

67. Torsional Strength:

For hollow shaft:

$$Pa = .1963 \frac{d^4 - d_1^4}{d} S$$

Pa = moment of the applied force.

d = external diameter of shaft.

d_1 = internal diameter of shaft.

S = unit shearing resistance.

LL	To d_1	Read d_1^4	To d_1	Read d_1^4			
C	Set 1	Opp. 4	Set 1	Opp. 4	Set d	Ind. to .1963	Opp. Ind.
D					To $d^4 - d_1^4$		Read Pa
CI						S to Ind.	

68. Radius of Gyration:

Spherical shell, radii R , r ; revolving on its dia.

$$G = .6325 \sqrt{\frac{R^5 - r^5}{R^3 - r^3}}$$

192 TYPICAL PROBLEMS AND SLIDE RULE SETTINGS

LL	To R	Read R ⁵	Read R ³	To r	Read r ⁵	Read r ³		
C or B	Set 1	Opp. 5	Opp. 3	Set 1	Opp. 5	Opp. 3		Index to .6325
A							To R ⁵ -r ⁵	
B							Set R ³ -r ³	
D								Read G

69. Flow of Air in Pipes:

$$Q = 3.287 \sqrt{\frac{pd^5}{L}}$$

Q = Quantity in cubic feet per second.

p = Head or pressure in lbs. per square inch.

d = Diameter in inches.

L = Length in feet.

LL	To d	Read d ⁵					
C or B	Set 1	Opposite 5	C			1 to d	Below 3.287
A				To d ⁵			
			B	Set L	Ind. to p		
			D			Read Q	Read Q

70. Work of Adiabatic Compression of Air:

$$\text{Mean effective pressure during the stroke} = 3.463 p_1 \left\{ \left(\frac{p_2}{p_1} \right)^{0.29} - 1 \right\}$$

Where p_1 and p_2 are absolute pressures above a vacuum in atmospheres or in pounds per square inch or per square foot.

Example: Required the work done in compressing one cubic foot of air per second from 1 to 6 atmospheres, including the work of expulsion from the cylinder.

$$\frac{p_2}{p_1} = 6$$

LL3	To 6					
C	Set 1	Opp. 29	Set 10	Below 3.463		At 10
LL2		Read 1.681				
D			To .681	Read 2.358 = atmos.	To 2.358	Read 34.66 lb. per sq. in. = mean effective pressure.
CI					Set 14.7	

(Continued)	CF	Above 144	Set 550	Above 1
	DF	Read 4990 lbs. per square foot × 1 foot stroke = 4990 ft. lbs.	To 4990	Read 9.08 HP

71. Airways:

To find the diameter of a round airway to pass the same amount of air as a square airway, the length and power remaining the same.

$$D^3 = \sqrt[5]{\frac{A^3 \times 3.1416}{.7854^3 \times O}}$$

where

D = the diameter of the round airway.

A = the area of the square airway.

O = the perimeter of square airway.

For slide rule purposes this may be written:

$$D = \sqrt[15]{\frac{A^3 \times 3.1416}{(.7854)^3 \times O}}$$

D	Under A	Under .7854		
K	Read A^3	Read $(.7854)^3$		Read $\frac{A^3 \times 3.1416}{(.7854)^3 \times O} = x$
DF			To A^3	
CF			Set $(.7854)^3$	
CI				Above O

(Continued)	LL	To x	Find D
	C	Set 15	Below 1

72. Flow of Steam in Pipes:

$$d = 0.5374 \sqrt[5]{\frac{Q^2 l}{h}}$$

where

d = internal diameter of pipe in inches.

Q = quantity of steam in cubic feet per minute.

l = length of pipe in feet.

h = height of a column of steam, of the pressure of steam at entrance, which would produce a pressure equal to the difference of pressures at the two ends of the pipe.

D	Ind. to Q				To $\sqrt[5]{\frac{Q^2 l}{h}}$	Read Ans.
B	h to Ind.	At l				
A		Read $\frac{Q^2 l}{h}$				
C			Set 5	At index	Set index	At .5374
LL			To $\frac{Q^2 l}{h}$	Read $\sqrt[5]{\frac{Q^2 l}{h}}$		

73. Effects of Bends and Curves in Pipes:

$$\text{Loss of head in feet} = \left\{ .131 + 1.847 \left(\frac{r}{R} \right)^{\frac{7}{2}} \right\} \times \frac{v^2}{64.4} \times \frac{a}{180}$$

r = internal radius of pipe in feet.

R = radius of curvature of axis of pipe in feet.

v = velocity in feet per second.

a = central angle, or angle subtended by bend.

C	Set R	Below index				
D	To r	Read $\frac{r}{R}$				
LL0			To $\frac{r}{R}$	Read $\left(\frac{r}{R} \right)^7$	To $\left(\frac{r}{R} \right)^7$	Read $\left(\frac{r}{R} \right)^{\frac{7}{2}}$
B			Set index	Above 7	Set 2	Above index

(Continued)	C	Set index	Below 1.847
	D	To $\left(\frac{r}{R} \right)^{\frac{7}{2}}$	Read $1.847 \left(\frac{r}{R} \right)^{\frac{7}{2}} = x$

(Continued)

D	Indicator to v			
B	Set 64.4 to Ind.	Ind. to .131 + x	180 to Ind.	Ind. to a
A				At Ind. read Ans.

74. Loss of Pressure and Head in Rubber-Lined Smooth 2½ Inch Hose:

$$P = \frac{lg^2}{4150 d^5} \quad h = \frac{lg^2}{1801 d^5}$$

P = pressure lost by friction in lbs. per sq. in.

l = length of hose in feet.

g = gallons of water discharged per minute.

d = diameter of hose in inches.

h = friction-head in feet.

C	Set index	Below 5		Indicator to g		
LL	To d	Read d ⁵				
A			To 1			Read Ans.
B			Set 4150 or 1801		d ⁵ to Indicator	At Index

75. Relation of Diameter of Pipe To Quantity of Water Discharged:

$$d = .239 \left\{ \frac{Q}{\left(\frac{h}{L} \right)^{\frac{1}{2}}} \right\}^{.387}$$

d = diameter of pipe in feet.

Q = quantity of water discharged in cubic feet per second.

h = head in feet.

L = length of pipe in feet.

A	To L					
B	Set h					
C		Below Q	Set Index	Under .387	Set index	Under .239
D		Find $\frac{Q}{\left(\frac{h}{L} \right)^{\frac{1}{2}}} = x$			To x ^{.387}	Read Ans.
LL			To x	Find x ^{.387}		

76. Density and Volume of Saturated Steam:

$$V = \frac{330.36}{p^{.941}} \quad D = \frac{p^{.941}}{330.36}$$

V = volume.

p = pressure in lbs. per square inch.

D = density.

C	Set Index	Below 941	Set $p^{.941}$	Below index	Read D
LL	To p	Read $p^{.941}$			
D			To 330.36	Read V	Opposite Index

77. Total Heat of Superheated Steam:

$$H = 0.4805(T - 10.38 p^{\frac{1}{2}}) + 857.2.$$

H = total heat of superheated steam.

T = temperature in degrees F. + 460.7.

p = pressure in lbs. per square foot.

C	Set 4	Below 1	Set 1	Below 10.38	Set 1
LL	To p	Read $p^{\frac{1}{2}}$			
D			To $p^{\frac{1}{2}}$	Read 10.38 $p^{\frac{1}{2}}$	To $T - 10.38 p^{\frac{1}{2}}$

(Continued)

C	Under 0.4805	$\left\{ \begin{array}{l} \text{To answer} \\ \text{add} \\ 857.2 \end{array} \right.$
LL		
D	Read .4805($T - 10.38 p^{\frac{1}{2}}$)	

78. Loss of Pressure of Steam Due to Friction:

Loss of power, expressed in heat units, due to friction:

$$H_f = \frac{W^3 f l}{10 p^2 d^5}$$

W = weight in lbs. of steam delivered per hour.

f = the coefficient of friction of the pipe.

l = length of pipe in feet.

p = absolute terminal pressure.

d = diameter of pipe in inches.

f is taken as from .0165 to .0175.

D	Under W				
K	Read W^3				
C		Set Index	At 5	Set p	
LL		To d	Read d^5		
A				To W^3	
B					Indicator to f

(Continued)	A		At Indicator read Answer.
	B	Set d^5 to Indicator	Indicator to 1

79. Mean Pressure of Expanded Steam:

$$P_m = p_1 \frac{1 + \log_e R}{R}$$

P_m = absolute mean pressure.

p_1 = the absolute initial pressure taken as uniform up to the point of cut-off.

R = absolute initial pressure.

C	Set Index	Read $\log_e R$	Set R	Indicator to p_1
LL	To e	Above R		
D			To $1 + \log_e R$	At Indicator read Ans.

80. Relative Efficiency of One Lb. of Steam With and Without Clearance:

Back pressure and compression not considered.

$$\text{Mean total pressure} = p = \frac{P(l + c) + P(l + c)\log_e R - Pc}{L}$$

P = initial absolute pressure in lbs. per sq. in.

$$R = \text{actual ratio of expansion} = \frac{L + c}{l + c}$$

l = period of admission measured from beginning of stroke.

c = clearance in inches.

L = length of stroke in inches.

C	Set $(l + c)$	At index	Set index	Indicator to $(l + c)$	Index to Ind.
D	To $(L + c)$	Read R			
LL			To R		

(Continued)	C	Indicator to P	Index to $(l + c)$	At P
	D	Read $P(l + c)\log_e R$		Read $P(l + c)$

(Continued)

C	Set L	At Index
D	To $P(l + c) + P(l + c)\log_e R - Pc$	Read p

81. Diameter of Piston Rods:

$$d = \sqrt[4]{\frac{D^2 p L^2}{a}} + \frac{D}{80}$$

D = diameter of cylinder in inches.

p = maximum unbalanced pressure in lbs. per sq. in.

L = length in feet.

a = 10,000 and upward, increasing with decrease in speed of engine.

C				At L	Set 4	At Index
A				Read $\frac{D^2 p L^2}{a}$		
B	Set a	Ind. to p	Index to Indic.			
D	To D					
LL					To $\frac{D^2 p L^2}{a}$	Read $\sqrt[4]{\frac{D^2 p L^2}{a}}$

(Continued)

C	Set 80	At index
A		
B		
D	To D	Read $\frac{D}{80}$
LL		

82. Journal Friction:

Coefficient when shaft is revolving

$$= (0.2 \text{ to } 0.3) \frac{\sqrt[5]{\text{vel. in ft. per min.}}}{\sqrt[5]{\text{press. in lbs. per sq. in.}}}$$

NOTE: This coefficient is for ordinary temperatures, pressures and speeds, with journals and bearing in good condition and well lubricated.

C	Set 5	Opposite index
LL	To vel. in ft. per min.	Find $\sqrt[5]{\text{vel. in ft. per min.}}$

(Continued)

C		At 0.2 or 0.3
D	To $\sqrt[5]{\text{vel. in ft. per min.}}$	Read Answer
A	Set press. in lbs. per sq. in.	

83. Storm Flow-Off:

$$\text{Cubic feet per second} = \frac{\text{A coefficient according to judgment}}{E} \times \frac{\text{Average cu. ft. of rainfall per sec. per acre during heaviest fall}}{N} \times \sqrt[4]{\frac{\text{Average slope of ground in feet per 1,000 feet} = S}{\text{No. of acres drained} = T}}$$

C	Set T	Opposite 1	Set 4	Opposite 1		At N
D	To S	Read $\frac{S}{T}$			To E	
LL			To $\frac{S}{T}$	Read $\sqrt[4]{\frac{S}{T}}$		
CI					Set $\sqrt[4]{\frac{S}{T}}$	Read Answer

84. Turbine Discs:

$$Y = Y_{a^e} \frac{-uw^2x^2}{2t} \quad \text{or} \quad Y = \frac{Y_a}{\frac{uw^2x^2}{e \cdot 2t}} \quad (\text{according to Stodola}).$$

Y = thickness at radial distance x

Y_a = thickness of disc carried to shaft center.

x = radial distance of a point from the axis.

u = specific mass.

w = angular velocity.

t = radial and tangential stress per unit of area.

C	Set index	Ind. to w	1 to Ind.	Ind. to x
A	To u			
B				

(Continued)	C				Read Ans.
	A				
	B	2 to Index	Ind. to Index	t to Ind.	At Index

Any number of values are thus found for $\frac{uw^2x^2}{2t}$, by giving x a

number of different values. With the indexes of C and LL coinciding, read under each value of $\frac{uw^2x^2}{2t}$ on C the corresponding value of $\frac{uw^2x^2}{e \cdot 2t}$ on LL.

From that point onward the solution is one of simple division on the C and D scales.

85. Catenary Curve:

$$y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$$

C	Set a	At Index	Set 1	At $\frac{x}{a}$		
D	To x	Read $\frac{x}{a}$			To e	
LL3			To e	Read $e^{\frac{x}{a}}$	Set 1	Read $e^{-\frac{x}{a}}$
LL0						At $\frac{x}{a}$

Since $e^{-\frac{x}{a}}$ is the reciprocal of $e^{\frac{x}{a}}$, the corresponding value of $e^{-\frac{x}{a}}$ may be found on CI directly over the value of $e^{\frac{x}{a}}$ on C.

The remainder of the computation is a simple addition; coupled with a simple division, using the CD scales.

Example: Let $a = 8$, and $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

x	$\frac{x}{a}$	$e^{\frac{x}{a}}$	$e^{-\frac{x}{a}}$	$e^{\frac{x}{a}} + e^{-\frac{x}{a}}$	$\frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) = y$
0	0.	1.	1.	2.	8.
1	.125	1.133	.882	2.015	8.06
2	.25	1.284	.779	2.063	8.252
3	.375	1.455	.687	2.142	8.568
4	.5	1.649	.606	2.255	9.02
5	.625	1.869	.535	2.404	9.616
6	.75	2.12	.472	2.592	10.368
7	.875	2.4	.416	2.816	11.264
8	1.	2.718	.368	3.086	12.344

86. Transmission Lines:

The capacity in Electro-static units of long transmission lines is given by:

$$C = \frac{l}{2 \text{ Log}_e \frac{l_1}{r}}$$

C	Set r	At index	Set l			Read answer
D	To l_1	Read $\frac{l_1}{r}$				At l
LL i			To e	Ind. to $\frac{l_1}{r}$		
CI					Set 2 to Ind.	

If length l is required the formula becomes

$$l = 2 C \text{ Log}_e \frac{l_1}{r}.$$

C	Set r	At index		At 2
D	To l_1	Read $\frac{l_1}{r}$		Read Answer.
LL			To $\frac{l_1}{r}$	
CI			Set C	

87. Hysteresis Loss:

$$W_h = n B^{1.6}$$

C	Set 1	At 1.6	Set 1	At n
LL	To B	Read $B^{1.6}$		
D			To $B^{1.6}$	Read $W_h = \text{Answer.}$

88. The Resistance of Dielectrics:

$$R = \frac{It}{C(\text{Log}_e E_1 - \text{Log}_e E_2)}$$

LL	At E_1	At E_2		
D	Read $\text{Log}_e E_1$	Read $\text{Log}_e E_2$	To It	Read R
C			Set C	
CI				At $(\text{Log}_e E_1 - \text{Log}_e E_2)$

89. Photometry and Light Transmission:

The general formula is as follows:

$$T' = T^{\frac{l'}{l}}$$

in which

T = transmission of the reference medium

l = length of path or thickness

T' = transmission sought for length l' .

(i) *Example:* A particular glass 2.4 cm. thick transmits, exclusive of the reflection loss, 88 per cent of a certain wave length. What will be the transmission of 6.4 cm. of the same glass?

$$T' = .88^{\frac{6.4}{2.4}}$$

LL0	To .88	Find .711. Answer
B	Set 2.4	Over 6.4

(ii) *Example:* A glass ray filter 3.2 mm. thick transmits, exclusive of the reflection loss, 52 per cent of violet light. What must be the thickness to transmit 75 per cent of the same light?

$$.75 = .52^{\frac{l'}{3.2}}$$

LL0	To .52	Opposite .75
B	Set 3.2	Find 1.4 mm. Answer

The light transmission measured by the photometer is the total, which includes the reflection loss at the two surfaces.

From photometric readings of total the relative transmission only can be computed by the formula.

$$T' = \left(\frac{T_1(n+1)^4}{16n^2} \right)^{\frac{l'}{l}}$$

In which

n = Refractive index of medium.

T' = Transmission sought for length l' .

T_1 = Total transmission of length l .

(iii) *Example:* A glass plate 1.5 cm. thick, refractive index 1.52, has a total transmission of 72 per cent of visible light. What will be the transmission only of the same glass, but 2.5 cm. thick?

$$T' = \left(\frac{.72 \times 2.52^4}{16 \times 1.52^2} \right)^{\frac{2.5}{1.5}}$$

A	To 7.2			Read .785		
B	Set 16				Set 1.5	At 2.5
C		Ind. to 2.52	1.52 to Ind.	At 2.52		
LL0					To .785	Find .668 Answer

CHAPTER V

SPECIAL FORMS OF THE SLIDE RULE

39. **Introduction.** — The slide rule described and explained in Chapter II is the basic standard form from which most other modern forms are derived. The rules treated in Chapter III are in general use in almost all the world, the names given in this book being those used in the United States. Beside these, which have become more or less standardized, there are many other forms of the slide rule which are specially constructed for certain purposes or have special scales for particular types of problems or calculations. A few of these will be described briefly in this chapter.

The various special forms of rules may of course take any desired construction form and they are actually of the greatest variety. In general, however, the aim seems to be to adhere to some more or less standard construction and to adapt the rule to its particular purpose by variations of the usual scale arrangements and the addition of new scales for the special calculations for which the rule is designed. In this way most of these special rules take the mechanical form of the Mannheim and duplex

rules, or they are made circular with scales on the circumferences or cylindrical with scales placed helically or parallel to the axis.

In what follows the rules discussed will be classified according to mechanical construction and two or three of each form, having different scales or scale arrangements, will be described.

Most of the rules described in this chapter carry two or more of the standard Mannheim scales which have already been described in detail. In the case of the other scales the special types of problems to which they are adapted limits their use to those who are particularly interested in such problems, or the special forms of their construction and scale division requires that a description, in order to be of value to the user, be given in considerable detail. In either case the general student and user will not be particularly interested in such descriptions, and the specially interested user will prefer to refer to the instructions issued with each form of rule and covering its operation in detail.

For these reasons the treatment given here will consist of a brief description of the construction and scales, accompanied in each case by a cut picturing the rule, and a brief statement of its uses, without any pretense to the fullness of the treatments given for the standard rules in Chapters II and III.

40. Slide Rules of Mannheim Form with Special Scales. — Two rules of this type will be described briefly; they are the Roylance Electrical Slide Rule and the Keuffel & Esser Stadia Slide Rule. The Roylance Electrical Rule is illustrated in Fig. 48 below which shows

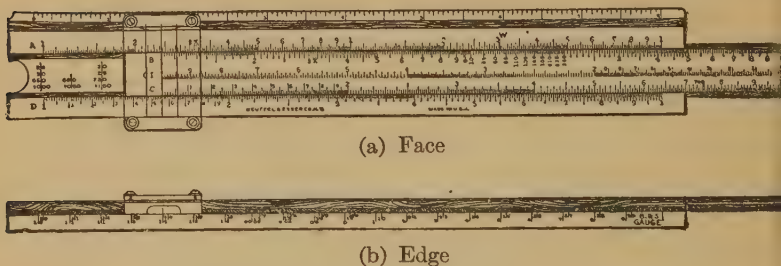


FIG. 48. ROYLANCE ELECTRICAL SLIDE RULE

the 8-inch rule. Mechanically this rule is exactly like the regular Mannheim rule and it has the same scales as the Mannheim rule and in addition the inverted (CI) scale of the polyphase rule.

On the regular C scale is a gauge point at 746 which is used for horse-power and kilowatt conversions. The regular B scale is complete, but the graduations beginning at 9.6 and ending at 20 serve also as a temperature scale in the determination of electrical resistance of wires at different temperatures. In this range the point 9.6 is specially marked, the middle index is marked in red, 11 is marked 25, 15 is marked 135, and 30 is marked 260. Consequently this portion of the B scale serves a double

purpose. On A2 near 3 is a gauge point marked "N" which is used in determining the weight per thousand feet of bare copper wire. In addition to the usual indicator hair line the runner carries two other such lines equidistant from it on either side. The distance between these outer lines is equal to the distance from the gauge point $\frac{1}{4}\pi = .7854$ to the right index on A2. Therefore if the runner is set with the right hand line at a diameter on D the circular area is at the left line on A. This device is useful in connection with wire size calculations.

The additional scales consist of a double scale on the lower edge of the stock, shown in Fig. 48 (b), and a multiple scale of gauge points on the stock under the slide, as shown in Fig. 48 (a). The scale on the edge gives the Brown & Sharpe gauge (American Wire Gage) numbers of wires when the central hair line of the runner is set on the diameter at D. The scales in the groove under the slide show the ampere carrying capacities of insulated wires and cables. The upper row of figures applies to rubber covered wire, the second row to weatherproof wire, the third to rubber covered cable and the fourth to weatherproof cable. The numbers on these scales are in line with the wire gauge scale divisions on the edge and are read in connection with those by means of the indicator. For wires the gauge numbers are regular B. & S. For cables they are in terms of 100,000 circular mils; thus

gauge No. 8 reads 800,000 c.m.; No. 14 reads 1,400,000 c.m., etc.

In addition to all the calculations possible with the Mannheim and many of those possible with the polyphase rules, therefore, the scales dealing with special electrical wiring problems make it possible to read directly at one setting of the indicator for any size of copper wire, the diameter in mils and inches, the section area in c.m. and sq. in., weight in pounds per thousand feet of bare wire, and resistance in ohms per thousand feet at any temperature in degrees Centigrade. In addition to these calculations and the power conversions, any of these data may of course be used in any calculation possible on the Mannheim rule or involving any of the standard electrical formulas.

The Keuffel & Esser Stadia slide rule, which is also of the same mechanical construction as the Mannheim rule,



FIG. 49. KEUFFEL & ESSER STADIA SLIDE RULE

is shown in Fig. 49. On the stock are two scales (see figure) both graduated the same as the standard A scale. The lower scale is marked to read 1 to 10 to 100 and indicated as "A." The upper one, indicated as "R,"

is marked to read 10 to 100 to 1000. On the back of the slide is a "B" scale like the "A" on the stock and when the slide is reversed these two are used together for the ordinary purposes of the C and D scales. The rule is made in 16 and 20 inch lengths and each half of "A" and "B" is thus equivalent to the 8 or 10 inch C or D scale.

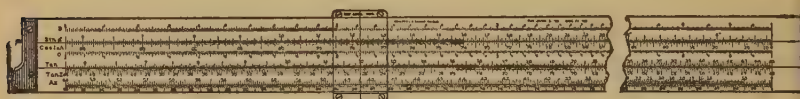
On the front of the slide, shown in the figure, are three scales called H, V and HC. The upper scale graduations are the same as those of the ordinary S scale and the markings are the same up to 40° . The remainder of this scale, instead of reading 50, 60, 70, 90° , reads from the right index toward the left as 0, 20, 30, 40, 45° . The left-hand portion is called "V" and the right-hand portion "H." The lower scale on the slide reads from left to right throughout its full length and is marked $4'$ to $5^\circ 43'$, which is the part of the angle scale (less than 45°) which does not appear on the ordinary T scale. This scale is also called "V." The third scale on the slide, between the upper and lower, is called "HC" and reads left to right from $1^\circ 49'$ to $18^\circ 15'$. These three scales of angles are used in connection with R and A on the stock to read the horizontal and vertical distances in stadia surveying when the vertical angle and stadia rod readings are known, by one setting of the slide.

This rule is graduated so that for the values of vertical

angle commonly met in stadia surveying the 20-inch rule gives the distances correct to the nearest 0.05 foot. The beveled edge of the projecting "lip" of the stock (shown in Fig. 11 (a)) is graduated in inches and tenths to serve as a plotting scale for distances.

41. Duplex Slide Rules with Special Scales. —

The duplex slide rule described in Article 27 lends itself particularly to special scale arrangements due to the large



(a) Front



(b) Back

FIG. 50. SURVEYOR'S DUPLEX SLIDE RULE

space available and to the fact that both faces are immediately accessible without the necessity of removing and reversing the slide. On this account duplex rules with many special forms of scale have been devised. Three such rules will be considered here.

The first of these is the Surveyor's Duplex Slide Rule, shown in Fig. 50. The back of this rule (Fig. 50 (b)) carries the regular A, B, C, D and CI scales and also the

H and V scales of the stadia slide rule described in the preceding Article.

The Front face (Fig. 50 (a)) carries the regular S and T scales (here marked "Sin" and "Tan") and also a cosine scale graduated in degrees. These are used in computing course latitudes and departures. On this face are five other scales arranged for the determination of the meridian from the transit readings of a direct solar observation. Since the operations involved in the use of these scales are somewhat complicated they will not be described here and the specially interested reader is referred to the book of instructions supplied with the rule. The rule is 20 inches long and when used in connection with the transit instrument with vertical circle and solar attachment, permits the making in the field of all the computations of surveying.

The second special form of duplex rule to be considered here is the Chemist's Duplex Slide Rule designed by Dr. R. Harmon Ashley. It is illustrated in Fig. 51.

On the front of the rule are the regular C and D scales in their usual positions and on the back are the CI and D scales. Beside the ordinary operations with these scales the direct and inverse proportions common in chemical calculations find them of particular convenience.

On the front of the rule are duplicate special scales or sets of gauge points giving the chemical formulas of the

more common acids, bases and salts; one is on the stock and one on the slide. On the back are similar and similarly located scales giving the symbols of the chemical elements and the formulas of the more common oxides. The gauge points on these four scales are arranged in such order and sequence that by using the scales directly in connection with one another the reaction products

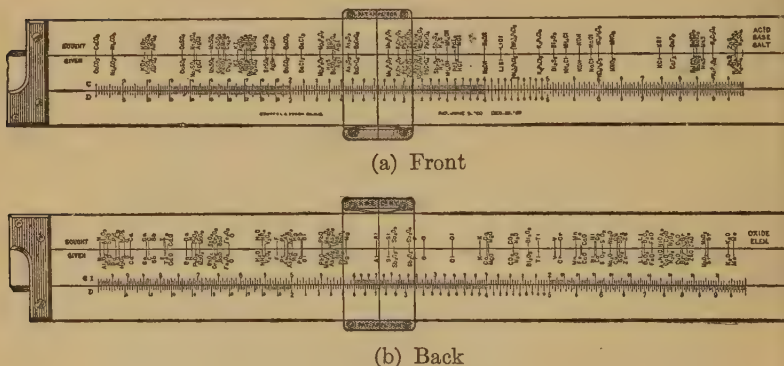
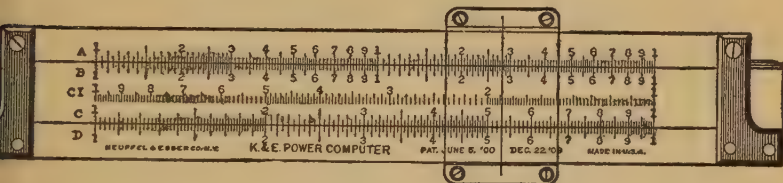


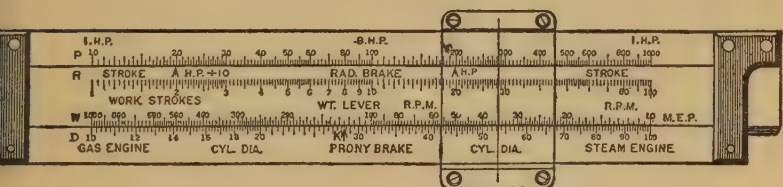
FIG. 51. ASHLEY CHEMIST'S SLIDE RULE

“sought” (see figure) may be found when the reagents are “given.” By using these scales in connection with C, D and CI the computations of gravimetric and volumetric analysis, stoichiometry, equivalents, percentage composition, volume of a gas from a given weight of substance at different temperatures and pressures, and other analogous problems, may be carried out by means of very simple operations.

A third type of duplex rule is the Power Computing Slide Rule (Fig. 52) designed especially for use in connection with the design, testing and operation of steam, gas and oil engines.



(a) Front



(b) Back

FIG. 52. POWER COMPUTING DUPLEX SLIDE RULE

The front face carries the regular A, B, C, D and CI scales and can therefore be used for all the ordinary forms of calculations.

On the back are the special scales P, R and W and a regular D scale which is marked 10 to 100. On all these scales are marked special gauge points and each scale is also marked to indicate the type of computation for which it is to be used. The computations for which these scales

and gauge points are specially adapted are those involving the power, steam or fuel consumption, dimensions and Prony brake tests of engines, and they also give certain standard data and conversion factors. Some of these computations are for speed, stroke, cylinder dimensions, brake and indicated horse-power, mean effective pressure, etc.

In ordinary calculations the position of the decimal point is usually known or obvious on inspection. In gas and steam engine work the scales as marked on this rule take care of the decimal point when the factors and results are within the numbered range. Any other values may be handled by shifting the decimal point as required. In Prony brake work, if all the factors taken are as numbered on the scales, the brake horse-power read at the H.P. index must be divided by 100.

42. Circular Slide Rules. — As seen in Chapter I the circular slide rule, having the scales laid off on concentric circular discs or rings of which one turns upon or within the other, was invented in England by William Oughtred before 1630 and independently reinvented in France by Jean Clairaut in 1716. The circular rule, while never very popular in England where most of the early development of the slide rule took place nor in the United States where most of the recent developments have

occurred, has since 1716 been much used in France and a widely used form of this type is the "Charpentier Calculator." The use of this rule is beginning to spread in the United States.

The Charpentier Slide Rule is shown in Fig. 53. It is of the general form of a watch, about $2\frac{1}{2}$ inches in diameter, and is made entirely of metal. The opposite faces are plane and it is in effect a single disc. The "stock" of the rule is a ring and the "slide" is a smaller disc which turns inside this disc with faces flush, being rotated on the center and set by means of a radial handle which carries at the end a "watch chain ring." On the front of the instrument (shown in the figure) is the C scale on the rotary slide and the D scale on the outer ring. The A and B scales in the usual double form are not included but an inner scale in two parts on the slide takes the place of the usual C scale ($\sqrt{1} = 1$ to $\sqrt{10} = 3.162$ to $\sqrt{100} = 10$) when the C scale is here used as the ordinary B scale. Thus the C scale and the two inner scales of this instrument are used together in the same way as the ordinary B and C scales.

On the back of the instrument are the usual L scale on the outer ring and the S and T scales on the disc slide. These are used in connection with the corresponding scales on the front by means of the pointers shown in the figure. The C and D scales are 6 inches in length, thus being

somewhat better than those of the ordinary 5-inch straight slide rule.

Fig. 54 illustrates another form of circular slide rule which is of American origin and known as the "Sperry Pocket Calculator." The instrument has the form of a watch with an engraved glass-covered metal dial on each



FIG. 53. CHARPENTIER CIRCULAR
SLIDE RULE



FIG. 54. SPERRY POCKET
CALCULATOR

side. Each dial has an index hand and a stationary pointer which together correspond to the runner (indicator) of the ordinary form of rule and which are set by the "stem" as in a watch. A second stem nut revolves the "slide" dial. The scales run twice around in the form of a spiral and the main logarithmic scales (C, D) are thus about $12\frac{1}{2}$ inches long, being therefore

somewhat better than the usual 10-inch rule. The instrument has all the Mannheim scales except S and T.

The advantages of these rules consist in their compactness and the fact that the scales are endless and there is no running off the scale and interchanging indexes as is the case with a straight rule.

43. Cylindrical Slide Rules. — The two cylindrical slide rules most used in the United States are the famous Thacher Rule (see Article 7) and Fuller's Slide Rule. These are illustrated in Figs. 55 and 56 respectively.

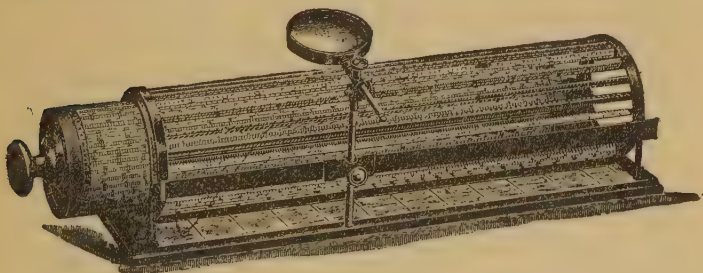


FIG. 55. THACHER CYLINDRICAL SLIDE RULE

As shown in the illustration, Fig. 55, the Thacher Slide Rule consists of a cylinder which is 4 inches in diameter and 18 inches long and which revolves in an open framework composed of 20 bars running lengthwise and held radially edgewise between two metal rings. The cylinder bears scales corresponding to the logarithmic scales of the

ordinary slide rule, which are duplicated on the exposed sides of the parallel bars. The cylinder slides and rotates inside the framework and any part of any one of its scales can be set at any part of any of the scales on the bars. The cylinder is provided with a knob at each end for sliding and rotating it, and the framework bears a 3-inch reading glass mounted on an adjustable stand which slides along the length of the instrument on a brass bar. The glass may thus be set at any part of the instrument and is also adjustable for focus.

The instrument has all the scales of the polyphase Mannheim slide rule and the equivalent total length is such that it may be read to four and five figures without the glass while with the glass it may be read to five and six figures. The scale arrangement is such that any of the usual slide rule operations involving as many as three given quantities are performed by one setting, while any number of values of an algebraic function composed of two constants and one variable may be found with a single setting.

The Thacher rule is by reason of its construction and range particularly suited to desk and drafting table use in engineering research and design offices and laboratories.

As shown in Fig. 56 Fuller's Slide Rule consists of a hollow outer cylinder (a) which slides and rotates on an inner cylinder (b), the inner cylinder being provided with

a handle (c). A single logarithmic scale nearly 42 feet long is wound helically on the outer cylinder and there are three indexes, one of which is on the fixed pointer (d) attached to the handle. A brass tube (e) slides inside the cylinder (b) and attached to it is a brass strip or bar (f) parallel to the axis. This bar carries two indexes

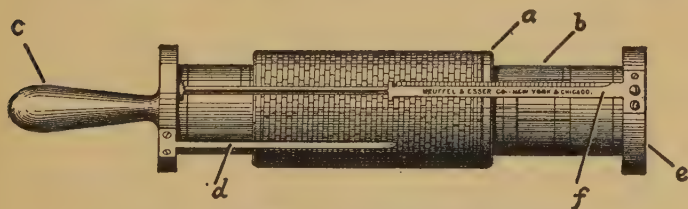


FIG. 56. FULLER CYLINDRICAL SLIDE RULE

whose distance apart is the axial length of the helical scale on (a), and a scale of equal parts for the finding of logarithms. On the cylinder (b) are printed a number of tables, gauge points and settings.

In using the ordinary slide rule any operation involving multiplication, division, powers and roots (special forms of multiplication and division), and proportion can all be looked upon as the solutions of proportions (see Articles 17 and 18). These proportions are set up on Fuller's Slide Rule by setting one number on the scale to the fixed index and the movable indicator arm to the second member on the scale, and then bringing the third member on the scale to the fixed index and reading the fourth on the

scale at the movable indicator. Hence the Fuller rule requires re-setting each time the third member of a proportion changes and it therefore does not give a series of equal ratios at sight in any one setting as do the C, D or A, B scales of the straight rule. Due to the extraordinary length of the scale however, it can be read with very great precision.

A cylindrical slide rule which has recently appeared takes the simple form of the popular and convenient refillable pencil. It carries only the C and D scales and these are arranged parallel to the axis of the pencil, one on the body and the other on a sliding sleeve. The pencil is about 6 inches long so that the rule is about equivalent to the ordinary 5-inch slide rule.

RECREATIONS IN MATHEMATICS

BY

H. E. LICKS

Copyright, 1917
By D. VAN NOSTRAND COMPANY

All Rights Reserved

*This book or any part thereof may not
be reproduced in any form without
written permission from the publishers.*

PRINTED IN THE U. S.

PREFACE

THE object of this book is to afford recreation for an idle hour and to excite the interest of young students in further mathematical inquiries. The topics discussed have therefore been selected with a view toward interesting students and mathematical amateurs, rather than experts and professors.

The Table of Contents is logically arranged with respect to chapters; but it will be found that within the latter, the topics are subject to no regular law or order. Some of these are long, others short; some are serious, others are frivolous; some are logical, others are absurd. It is feared that many things which might have been included have been omitted, and that still others which should have been omitted, have been included. The indulgence of readers is craved for this seeming lack of consistency and it is submitted in extenuation that the very character of the subject, partaking as it does somewhat of the nature of the curio collection, renders a more orderly treatment practically impossible.

The subject matter has been collected from many and divers sources and it is hoped that in spite of the complex nature of the work, the selection will appeal to the readers to whom it is addressed.

H. E. L.

December, 1916.

CONTENTS

CHAPTER	PAGE
I. ARITHMETIC.....	I
II. ALGEBRA.....	22
III. GEOMETRY.....	44
IV. TRIGONOMETRY.....	63
V. ANALYTIC GEOMETRY.....	76
VI. CALCULUS.....	87
VII. ASTRONOMY AND THE CALENDAR.....	104
VIII. MECHANICS AND PHYSICS.....	124
IX. APPENDIX.....	140

RECREATIONS IN MATHEMATICS

CHAPTER I

ARITHMETIC

1



COUNTING a series of things and keeping tally of the tens on the fingers were processes used by primitive peoples. From the ten fingers arose ultimately the decimal system of numeration. Recording the results of counting was done by the Egyptians and other ancient nations by means of strokes and hooks; for one thing a single stroke **I** was made, for two things two strokes **II** were used, and so on up to ten which was represented by **∩**. Then eleven was written **I∩**, twelve **II∩**, and so on up to twenty, or two tens, which was represented by **∩∩**. In this way the numeration proceeded up to a hundred, for which another symbol was employed.

Names for **II**, **III**, **IIII**, **∩∩**, etc., appear in the Egyptian hieroglyphics, but a special symbol for each name is not used. Probably the Hindoos first invented such symbols, and passed them on to the Arabs, through whom they were introduced into Europe.

2

GREEK NOTATION

The Greeks used an awkward notation for recording the results of counting. The first nine letters of the Greek alphabet denoted the numbers from one to nine, so that α

represented one, β two, γ three, and so on. Then the following nine letters were used for ten, twenty, thirty, etc., so that κ represented ten, λ twenty, μ thirty, and so on. Then the remaining letters τ , υ , etc., were used for one hundred, two hundred, etc., but as the Greek alphabet had only twenty-four letters, three symbols were borrowed from other alphabets. This was an awkward notation, and there seems to have been little use made of it except to record results. A number having two letters was hence between ten and one hundred, and one having three letters was between one hundred and one thousand; thus, $\lambda\delta$ was twenty-four and $\tau\kappa\delta$ was one hundred and fourteen. The Greeks were good mathematicians, as appears from their work in geometry, but only a few writers used this arithmetical numeration in computations, saying, for example, that the sum of $\kappa\alpha$ and $\lambda\beta$ was $\mu\gamma$. In those days the abacus or swan pan, similar to that seen in Chinese laundries in the United States, was employed to make arithmetical computations. From very early days this simple apparatus has been used throughout the East, and it is said that computations are made on it with great rapidity.

3

ROMAN NUMERATION

The Romans represented the first five digits by I, II, III, IIII, and V, a V prefixed to the first four gave the digits from six to nine, while ten was represented by X, fifty by L and one hundred by C. This notation is still in use for a few minor purposes, it being modified by using IV for four, IX for nine, etc.; when a watch face is lettered in this notation, however, IIII is always used for IV, because Charles V said that he would allow nothing to precede a V. The

Roman notation was employed only to record numbers, and it does not appear that arithmetical operations were ever conducted with it. Perhaps this awkward notation retarded the development of mathematics among the Romans.

Frontinus, a Roman water commissioner, wrote in 97 A.D. a treatise on the Water Supply of the City of Rome, a translation of which, with an excellent commentary by Clemens Herschel, was published at Boston in 1889. A long list of the dimensions of the water pipes then in use is given, these being expressed in digits and fractions. The fraction $1/12$ was denoted by a single horizontal stroke —, $2/12$ by two strokes ==, $3/12$ by three strokes ===, and so on up to $5/12$. Then $1/2$ was represented by *S*, while the fractions from $7/12$ to $11/12$ were represented by adding strokes to the *S*, thus, $S==$, indicated $1/2 + 4/12$ or $5/6$. The fraction $1/24$ was indicated by *ℒ*. The smallest fraction used was $1/288$ which was represented by *Θ*. The following is the description of the pipe No. 50 given by Frontinus:

Fistula quinquagenaria: diametri digitos septem S==— ℒΘ quinque, perimetri digitos XXV ℒΘ VII, capit quinaras XLS = ℒΘV.

Of which Herschel's translation is as follows:

The 50-pipe: seven digits, plus $1/2$, plus $5/12$, plus $1/24$, plus $5/288$ in diameter; 25 digits, plus $1/24$ plus $7/288$ in circumference; 40 quinaras, plus $1/2$, plus $2/12$, plus $1/24$, plus $5/288$ in capacity.

The digit was one-sixteenth of a Roman foot and the quantity of water flowing through a pipe of $1\frac{1}{4}$ digits in diameter was called a quinaria. Frontinus takes the quantities of water flowing through pipes as proportional to the squares of their diameters, for he says that pipes of $2\frac{1}{2}$ and $3\frac{3}{4}$ digits in diameter discharge four and nine quinaras respectively.

4

THE ARABIC SYSTEM OF NUMERATION

The Arabic method, by which the symbols, 1, 2, 3, etc., were used for the first nine integers, seems to have first originated in India, from whence it was carried by the Arabs to Europe, about the year 1200.

Long before this time Greek and Arabic astronomers had used the sexagesimal system in the division of the circle, and this, with Arabic numerals, was employed about 1200 in Europe for expressing numbers not at all connected with a circle. Thus, $28, 32' 17'' 45''' 20^{IV}$ meant 28 units plus $32/60$, plus $17/3600$, plus $45/216000$, plus $20/1296000$. This method of expressing fractions was certainly more convenient than the Roman method as used by the water commissioner Frontinus.

How the Arabic method of numeration was introduced into Europe is told by Ball in the following interesting account of one of the early Italian mathematicians.

5

LEONARDO DE PISA

From Ball's Short Account of the History of Mathematics. Fourth Edition (London, 1908), pages 167-170.

Leonardo Fibonacci (i.e., filius Bonacci), generally known as Leonardo of Pisa, was born at Pisa about 1175. His father Bonacci was a merchant, and was sent by his fellow-townsmen to control the custom-house at Bugia in Barbary; there Leonardo was educated, and he thus became acquainted with the Arabic or decimal system of numeration, as also with Alkariami's work on Algebra. It would seem that Leonardo was entrusted with some duties, in connection

with the custom-house, which required him to travel. He returned to Italy about 1200, and in 1202 published a work called *Algebra et almuchabala* (the title being taken from Alkariami's work), but generally known as the *Liber Abaci*. He there explains the Arabic system of numeration, and remarks on its great advantages over the Roman system. He then gives an account of algebra, and points out the convenience of using geometry to get rigid demonstrations of algebraical formulas. He shows how to solve simple equations, solves a few quadratic equations, and states some methods for the solution of indeterminate equations; these rules are illustrated by problems on numbers. The algebra is rhetorical, but in one case letters are employed as algebraical symbols. This work had a wide circulation, and for at least two centuries remained a standard authority from which numerous writers drew their inspiration.

The *Liber Abaci* is especially interesting in the history of mathematics, since it practically introduced the use of Arabic numerals into Christian Europe. The language of Leonardo implies that they were previously unknown to his countrymen: he says that having had to spend some years in Barbary he there learnt the Arabic system, which he found much more convenient than that used in Europe; he therefore published it "in order that the Latin race might no longer be deficient in that knowledge." Now Leonardo had read very widely, and had travelled in Greece, Sicily, and Italy; there is therefore every presumption that the system was then not commonly employed in Europe.

The majority of mathematicians must have already known of the system from the works of Ben Ezra, Gerard, and John Hispalensis. But shortly after the appearance of Leonardo's book Alfonso of Castile (in 1252) published

some astronomical tables, founded on observations made in Arabia, which were computed by Arabs, and which, it is generally believed, were expressed in Arabic notation. Alfonso's tables had a wide circulation among men of science, and probably were largely instrumental in bringing these numerals into universal use among mathematicians. By the end of the thirteenth century it was generally assumed that all scientific men would be acquainted with the system; thus Roger Bacon writing in that century recommends algorism (that is, the arithmetic founded on the Arab notation) as a necessary study for theologians who ought, he says, "to abound in the power of numbering." We may then consider that by the year 1300, or at the latest 1350, these numerals were familiar both to mathematicians and to Italian merchants.

So great was Leonardo's reputation that the Emperor Fredrick II stopped at Pisa in 1225 to test Leonardo's skill, of which he had heard such marvellous accounts. The competitors were informed beforehand of the questions to be asked, some or all of which were composed by John of Palermo, who was one of Fredrick's suite. This is the first time that we meet with an instance of those challenges to solve particular problems which were so common in the sixteenth and seventeenth centuries. The first question propounded was to find a number of which the square, when either increased or decreased by five, would remain a square. Leonardo gave an answer, which is correct, namely $41/12$. The other competitors failed to solve any of the problems. (See No. 33 for a problem in Algebra.)

6

EARLY ARITHMETIC IN ENGLAND

The earliest book on arithmetic printed in England was "The Grounde of Artes, by M. Robert Recorde, Doctor of Physik." First issued in 1540 it was republished in numerous editions until 1699. The following extracts from the edition of 1573 give an idea of the method of instruction.

Master. — If numbering be so common that no man can doe anything alone, and much less talke or bargain with other, but still have to doe with numbre; this proveth not numbre to be contemptible and vile, but rather right excellent and of high reputation, sithe it is the grounde of all mens affaires, so that without it no tale can be told, no bargaining without it can dully be ended, or no business that man hath, justly completed. . . . Wherefore in all great workes are Clerkes so much desired? Wherefore are Auditors so richly feed? What causeth Geometricians so highly to be enhaunced? Because that by numbre suche things they find, which else would farre excell mans minde.

Scholar. — Verily, sir, if it be so that these men by numbring their cunning doe attaine, at whose great workes most men doe wonder, then I see well that I was much deceived, and numbring is a more cunning thing than I take it to be.

Master. — If numbre were so vyle a thing as you did esteem it, then need it not to be used so much in mens communication. Exclude numbre, and answer to this question: How many years old are you?

Scholar. — Mum.

Master. — How many daies in a week? How many weeks in a yeare? What landes hath your father? How many men doth he keep? How long is it sythe you came from him to me?

Scholar. — Mum.

Master. — So that if numbre wante, you answer all by Mummies.

The master then goes on to show how useful numbers are in "Musike, Physike, Law, Grammer" and such like, and then proceeds to teach him numeration, addition, subtraction, and so on. The master explains and illustrates the process and then tests the scholar by requiring him to

perform an example, the latter explaining as he goes on and asking questions on doubtful points. Thus, in addition, after having explained the process of carrying, the master gives the scholar the numbers 848 and 186 to be added.

Scholar. — I must set them so, that the two first figures stand one ouer another, and the other each ouer a fellow of the same place. And so likewise of other figures, setting always the greatest numbre highest, thus, as followeth:

$$\begin{array}{r} 848 \\ 186 \\ \hline \end{array}$$

Then I must add 6 to 8 which maketh 14, that is mixt numbre, therefore must I take the digit 4 and write it under the 6 and 8, keeping the article 1 in my mind, thus:

$$\begin{array}{r} 848 \\ 186 \\ \hline 4 \end{array}$$

Next that, I doe come to the second figures, adding them up together, saying 8 and 4 make 12, to which I put the 1 reserved in my mind, and that maketh 13, of which numbre I write the digit 3 under 8 and 4, and keep the article 1 in my mind, thus:

$$\begin{array}{r} 848 \\ 186 \\ \hline 34 \end{array}$$

Then come I to the third figures, saying 1 and 8 make 9, and the 1 in my mind maketh 10. Sir, shall I write the cypher under 1 and 8?

Master. — Yea.

Scholar. — Then of the 10 I write the cypher under 1 and 8 and keep the article in my mind.

Master. — What needeth that, seeing there followeth no more figures?

Scholar. — Sir, I had forgotten, but I will remember better hereafter. Then seeing that I am come to the last figures, I must write the cypher under them, and the article in a further place after the cypher, thus:

$$\begin{array}{r} 848 \\ 186 \\ \hline 1034 \end{array}$$

Master. — So, now you see, that of 848 and 186 added together, there amounteth 1034.

Scholar. — Now I think I am perfect in addition.

Master. — That I will prove by another example. There are two armies: in the one there are 106 800 and in the other 9400. How many are there in both armies, say you?

In those old days it seems that the multiplication table was learned only as far as five times five, and hence a process was necessary for multiplying together two numbers like 6 and 8. The following is the process as given by Recorde. The numbers 6 and 8 were placed on the left-hand side of a large letter X, thus:

$$\begin{array}{c} 8 \\ \diagdown \quad \diagup \\ 6 \end{array}$$

Then each was subtracted from 10, the remainders placed directly opposite on the right-hand side, and a line was drawn under the whole, thus:

$$\begin{array}{c} 8 \quad \quad 2 \\ \diagdown \quad \diagup \\ 6 \quad \quad 4 \\ \hline \end{array}$$

Next the units figure of the product was found by multiplying together the remainders 2 and 4, and the other figure of the product by subtracting crossways either 2 from 6 or 4 from 8; thus, 2 times 4 is 8, and 2 from 6 (or 4 from 8) is 4; therefore,

$$\begin{array}{c} 8 \quad \quad 2 \\ \diagdown \quad \diagup \\ 6 \quad \quad 4 \\ \hline 48 \end{array}$$

and hence six times eight makes forty-eight.

Napier's bones, used in England in the seventeenth century, consisted of nine sticks numbered at the top 1 to 9 inclusive, each stick having on its side the first nine multiples of the number at the top. These bones were hence

merely a multiplication table. When it was desired to multiply a number by 57, the sticks headed 5 and 7 were taken and their multiples used. Thus suppose that 89 was to be multiplied by 57. First, looking on the stick 5, the

$$\begin{array}{r}
 89 \\
 \underline{57} \\
 40 \\
 45 \\
 56 \\
 \underline{63} \\
 5073
 \end{array}$$

multiples of 5 by 8 and 9 were taken off and set down as shown, then looking on the stick 7 the multiples of 7 by 8 and 9 were taken off and set down; then the addition gave the product of 89 by 57. Thus were arithmetical operations performed in England less than four hundred years ago.

7

THE SIGNS OF ARITHMETIC

The signs $+$ and $-$ are supposed to have been first used in Holland in the fifteenth century, to denote excess or deficiency in weight of bales of goods. The normal weight of a certain bale being, say, 4 centners, it was marked 4 c. $+$ 5 lb. if it weighed 5 lb. more than the normal, and 4 c. $-$ 5 lb. if it weighed 5 lb. less. These signs were used in a similar sense in Widman's Arithmetic published at Leipzig in 1489. It was not until about 1540 that they were used as signs of operation, that is, as directions to perform addition or subtraction.

The sign $=$ was first used in works on Algebra, the earliest

mention being in Recorde's Whetstone of Wit issued in 1557, his selection of that sign being because "no 2 thyngs can be moare equalle."

The decimal point came much later, for fractional numbers were generally written in the duodecimal or sexagesimal form prior to the fifteenth century, as has already been explained. Napier and Briggs, the inventors of logarithms seem to have been the first to use, about 1620, the decimal method and the decimal point, although at first there was no point, but a line was drawn under all the decimals.

It is scarcely more than a hundred years since the decimal point came to be generally used in the United States. For example, Willett's Scholar's Arithmetic, used in the public schools of New York City was issued in a fourth stereotype edition in 1822. On page 23 it is said that in adding sums of money, the dollars, cents, and mills should be kept separate by placing a point between them, but the point used is a colon. On page 24 in subtraction, it is said that dots must be used to keep these units separate, but the dot used is a comma. Under multiplication the colon is used in some examples and the comma in others. Under division (page 27) the comma is used, and also numbers like \$56.43 are written \$56 43cts. Under "Reduction of Federal Money" on page 55, the single parenthesis is used as a decimal point; the problem being to reduce 387652 mills to dollars, the number is first divided by ten and the result stated as 38765(2; then this is divided by 100 giving 387(65:2, and finally the answer is given as 387 65cts 2m. Nothing more is said about decimals until we come to "Decimal Fractions" on page 151, and there the period is formally introduced as the decimal point, and the operations on numbers containing decimals are well explained;

even then it seems necessary to mention that a quantity like \$590.217 means 590 dols. 21 cts. 7 m. A large part of the time of the children who used this book was devoted to intricate problems concerning pounds, shillings, pence, and farthings.

The signs $\times$ and $\div$ to indicate the operations of multiplication and division were not in common use before 1750. Prior to this date parentheses were not used in a case like $a(b + c + d)$, but a straight line was drawn over the $b + c + d$.

The use of the shilling mark / to indicate division is comparatively recent, it having been first employed about 1860. In this country it was rarely used until after 1890, but is now very commonly found in algebraic notation, and it will generally be used in the later chapters of this volume.

Thus, $4/261$ has the same meaning as $\frac{4}{261}$ or $4 \div 261$.

This new division mark is of especial advantage in simplifying printed work, either in setting algebraic expressions in type or in writing fractions with a typewriter.

ARITHMETIC AMUSEMENTS

8

Multiply 37037037 by 18; also multiply it by 27.

Multiply 1371742 by 9; also multiply it by 81.

Multiply 98765432 by 9; also multiply it by $1\frac{1}{8}$.

9

Think of any number, multiply it by 2, then add 4, multiply by 3, divide by six, subtract the number you thought of, and the result will be 2.

10

Think of any number, and add 1 to it, then square these two numbers and subtract the less from the greater. Now if you will tell me this difference, I can easily know the number you thought of, for I merely subtract 1 from the number you give me, then divide by 2, and the result is the number of which you thought.

11

To find the age of a man born in the nineteenth century. Ask him to take the tens digit of his birth year, multiply it by ten and add four; to this ask him to add the units figure of his birth year and tell you the result. Subtract this from 124 and you will have his age in 1920. Thus for a man born in 1848, $4 \times 10 = 40$, $40 + 4 = 44$, $44 + 8 = 52$, $124 - 52 = 72$, which will be his age in 1920, if then living.

12

Ask a person to multiply his age by 3, add 6 to the product, then divide the last number by three and tell you the result. Subtract two from that result and you have his age.

13

A woman goes to a well with two jars, one of which holds 3 pints and the other 5 pints. How can she bring back exactly 4 pints of water?

14

By the help of the following table the age of a person under 21 can be ascertained:

	A	B	C	D	E
Ask the person to tell you in	1	2	4	8	16
which column his or her age	3	3	5	9	17
occurs. Then add together the	5	6	6	10	18
numbers at the tops of those	7	7	7	11	19
columns and the sum will be the	9	10	12	12	20
age. Thus, if a person says that	11	11	13	13	
his age is found in columns B and	13	14	14	14	
E, then $2 + 16 = 18$ which is	15	15	15	15	
his age.	17	18	20		
	19	19			

15

All integral numbers are either prime or composite. A prime number is one which has no integral divisors except itself and unity. There is no simple method of ascertaining what numbers are prime except that of the "sieve of Eratosthenes." By this method the odd integral numbers are written in ascending order, thus, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, etc., then every third number after three, every fifth number after 5, every seventh number after 7, and so on, are crossed off, and those remaining are primes. Thus from the above numbers, 9, 15, 21, and 27 are first crossed off, then 15 and 25, and then 21; the remaining numbers 3, 5, 7, 11, 13, 17, 19, 23, 29, together with 2, are the prime numbers less than thirty. This process becomes very laborious when the numbers run into millions.

16

A perfect number is one which is equal to the sum of its divisors. Thus, 6 equals $1 + 2 + 3$ and 1, 2, and 3 are the divisors of 6. The next perfect number is 28, the third one

is 496, and the fourth one is 8128. Beyond this there are no perfect numbers until 33 550 336 is reached. Then come 8 589 869 056, 137 438 691 328 and 2 305 843 008 139 952 128. This seems to be the largest perfect number thus far found. All of the above numbers end in 6 or 28. It is not known whether or not an odd number can be perfect, but the indications are against this being the case. The above numbers are taken from Ozanam's *Recréations Mathématiques*, published at Paris in 1750. Other numbers which he gives are not perfect, because he unfortunately made errors in computing them. Ozanam's book was first published in 1698; it passed through many editions and was also translated into English.

17

At rare intervals natural calculating boys come to public notice. One of these was Zerah Colburn who was born in New England in 1804 and taken to London when eight years old to exhibit his powers. He could mentally multiply any number less than 10 into itself successively nine times and give the results faster than they could be written down. He was asked what number multiplied by itself gave 106 989 and he instantly replied 327. With equal promptness he stated that the number which multiplied twice by itself gave 268 336 125 was 645, this being a problem in cube root for which an ordinary computer would require several minutes. He was asked to name a number which would divide exactly 36 083 and he immediately replied that there was no such number, in other words he recognized this as a prime number just as readily as we recognize 29 or 37 to be one. He could very quickly multiply together two numbers of four or five figures, and perform

many other remarkable mental feats. These natural calculators are rarely able to explain their processes, and their powers fade away and disappear as they grow up and become educated.

18

How many people know that the square of 3 plus the square of 4 equals the square of 5? All surveyors and draftsmen know it, also most machinists and carpenters, but to those in other trades it is probably quite unknown.

Another interesting arithmetical theorem is that the cube of 3 plus the cube of 4 plus the cube of 5 equals the cube of 6. Probably few students who read this book have ever before heard of this important relation.

19

In one hand a person has an odd number of coins or pebbles and in the other hand an even number, the knowledge of the same being unknown to anyone except himself. Ask him to multiply the number in the right hand by 2 and the number in the left hand by 3. Then ask him to add together the two products and tell you their sum. If this sum is odd the left hand has the odd number of coins, but if the sum is even, the left hand has the even number of coins.

20

One tumbler is half full of wine and another tumbler is half full of water. A teaspoonful of wine is taken from the first tumbler and put into the other one. Then a teaspoonful of the mixture is taken from the second tumbler and put into the first one. Is the quantity of wine removed from the first tumbler greater or less than the quantity of water removed from the second tumbler? Ball in his *Mathe-*

matical Recreations and Essays says that the majority of people will say it is greater, but that this is not the case. H. E. Licks, who has studied this problem, claims that the two quantities are exactly equal.

21

A stranger called at a shoe store and bought a pair of boots costing six dollars, in payment for which he tendered a twenty-dollar bill. The shoemaker could not change the note and accordingly sent his boy across the street to a tailor's shop and procured small bills for it, from which he gave the customer his change of fourteen dollars. The stranger then disappeared, when it was discovered that the twenty-dollar note was counterfeit, and of course the shoemaker had to make it good to the tailor. Now the question is, how much did the shoemaker lose?

22

At an humble inn where there were only six rooms, seven travellers applied for lodging, each insisting on having a room to himself. The landlord put the first man in room No. 1 and asked one of the other men to stay there also for a few minutes. He then put the third man in room number two, the fourth man in room No. 3, the fifth man in room No. 4, and the sixth man in room No. 5. Then returning to room No. 1 he took the seventh man and put him in room No. 6. Thus each man had his own room!

23

An Arab merchant directed by will that his seventeen horses should be divided among his three sons, one-half of them to the eldest, one-third to the second son and one-ninth to the youngest son. How to make the division was

a serious problem, for the eldest son claimed nine horses, but the others objected because this was more than one-half of seventeen. In this dilemma they applied to the Sheik who put his white Arabian steed among the seventeen horses, directed the eldest son to take one-half of the eighteen or nine, the second son to take one-third of the eighteen or six, and the youngest son to take one-ninth of the eighteen or two. Thus, since nine plus six plus two are seventeen, the horses were divided satisfactorily among the three sons. "Now," said the Sheik, "will I take away my own horse," and he led the Arabian steed back to his peg in the pasture.

24

To add 5 to 6 in such a way that the sum may be 9. Make six marks at equal distances apart, thus // // // // // /. Between the first and second marks draw a slanting line so as to form the letter *N*; then do the same between the fourth and fifth marks; finally add to the last line three horizontal marks so as to form the letter *E*. Then the problem is solved, for the five marks added to the given six marks have made *NINE*.

Another interesting problem in this line is to add three marks to a given five so as to make a quotation from Shakespeare. The added three marks give *KINI*, and you ask, where in this is found the required quotation. After a few minutes silence I reply, "A little more than kin but less than kind."

25

Two impossible problems: (1) If 3 is one-third of 10, what is one-quarter of twenty? (2) A man who had a bale of cotton sold it for \$50, bought it back for \$45, and then sold it again for \$65. What was the net gain to the man?

26

The Indian mathematician Sessa, the inventor of the game of chess, was ordered by the king of Persia to ask as a recompense whatever he might wish. Sessa modestly requested to be given one grain of wheat for the first square of the board, two for the second, four for the third, and so on, doubling each time up to the sixty-fourth square. The wise men of the king added the numbers 1, 2, 4, 8, 16, etc., and found the sum of the series to sixty-four terms to be 18 446 744 073 709 551 615 grains of wheat. Taking 9000 grains in a pint we find the whole number of bushels to be over 32 000 000 000 000, which is several times the annual wheat production of the whole world.

27

H. E. Licks once had a class of students well versed in arithmetic, algebra, trigonometry, and calculus, but not one of them could solve the following simple problem, as they knew nothing about bookkeeping. The problem is hence here given for other young people.

A Coal Company appointed an agent, agreeing to pay him a salary of \$265 for six months, all of the coal at the end of that time and all of the profits to belong to the Company. The Company furnished him with coal to the amount of \$825.60 and in cash \$215.00. The agent received for coal sold \$1323.40, paid for coal bought \$937.00, paid sundry expenses authorized by the Company \$129.00, paid his own salary \$265.00, paid to the Company \$200.00, delivered to indigent persons by order of the Company coal to the amount of \$13.50. At the end of the six months the Company took possession and found coal amounting to

\$616.50. The agent then paid to the Company the money belonging to them. How much did he pay? Did the Company gain or lose by the agency and how much?

28

THE FIFTEEN PUZZLE

About the year 1880 everyone in Europe and America was engaged in the solution of this interesting puzzle. A square shallow box contained fifteen blocks numbered 1 to 15 inclusive and these could be moved about one block at a

8	14	3	5
10	11	9	1
4	6		2
13	7	15	12

Fig. 1.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Fig. 2.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Fig. 3.

time, on account of the blank space. The blocks being placed in the box at random, say as shown in Fig. 1, the problem was to arrange them in regular order in the manner shown in Fig. 2. It was a fascinating exercise to shift these blocks until 1 was brought to the upper left-hand corner, then to bring 2 next to it, and thus keep on until the regular order of Fig. 2 was secured. But sometimes it happened, when the lowest row was reached, that the order of Fig. 3 resulted; for this case mathematicians proved that it was impossible to cause the blocks to take the regular order of Fig. 2. Mathematical analysis also showed that for many random positions of the blocks (Fig. 1), one-half of them would result in the order of Fig. 2 and one-half in the order of Fig. 3.

There is, however, a way by which the arrangement of Fig. 3 can be brought into regular order. Move the blocks until the upper left-hand corner is blank and the blocks 1, 2, 3 fill the other spaces of the upper row, then continue until the blocks 4, 5, 6, 7 fill the second row and 8, 9, 10, 11 the third row, then the lowest row can be arranged in the order 12, 13, 14, 15. This is a solution of the puzzle, if the statement of the problem is merely that "the blocks are to be arranged in regular order."

This puzzle comes under that branch of arithmetic known as permutations and combinations, and much mathematical thought has been expended upon it. The number of ways in which the fifteen blocks can be put at random in the box (Fig. 1) is 1 307 674 368 000. In the early stages of the craze it was not recognized that half of these combinations lead to the result of Fig. 2 and half of them to the result of Fig. 3. Hence when Fig. 3 was reached, the player usually kept on trying to obtain the arrangement of Fig. 2. Finally, after mathematicians had proved that it is impossible to bring Fig. 3 to agree with Fig. 2, the craze abated.

It has been stated that this interesting puzzle was invented in 1878 by a deaf and dumb man as a solitaire game. In the height of the craze persons in public conveyances could be seen every day attempting to solve the puzzle. Some physicians thought that this work was a beneficial mental exercise, but others claimed that it led to nervous disorders. A poet well expressed the latter opinion as follows:

Put away his crack-brain puzzle,
He has climbed the asylum stair;
Numbers thirteen, fifteen, fourteen,
Turned his head and sent him there!

CHAPTER II

ALGEBRA

29



UCLID used algebra in a geometric form, expressing the equations always by words. For example: if a straight line be divided into two parts, the square on the line is equal to the sum of the squares on the two segments plus twice the rectangle of those segments; this in modern algebra is the theorem $(a + b)^2 = a^2 + b^2 + 2ab$. Until the sixteenth century all algebraic equations were generally expressed in words; for instance, Omar Kayyam wrote about 1100, "Cubus, latera et numerus aequalis sunt quadratis," meaning $x^3 + bx + c = ax^2$. Cardan about 1550 wrote "Cubus $\bar{p}$ 6 rebus aequalis 20," meaning $x^3 + 6x = 20$. Ramus about the same time wrote $7q + 3l - 2$ where l meant the unknown quantity and q its square. At this date $R_3 17$ was introduced which later became $\sqrt[3]{17}$; here the R signified radix or root.

Vieta, the founder of modern algebra, wrote about 1580 $1C - 8Q + 16N$ aequ 40, signifying $x^3 - 8x^2 + 16x = 40$, since C meant cube, Q meant square and N meant first power of the unknown quantity. Fifty years later Descartes introduced x to represent the unknown quantity in the equation. Algebra as we now know it is scarcely more than three hundred years old.

30

The earliest note of an equation is found in the Egyptian records of Ahmes in the following form "heap, its two-thirds, its half, its seventh, its whole gives 97," that is, heap is the unknown quantity x , and $\frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x + x = 97$ is the equation to be solved. It was a long time, however, before equations of the second degree appeared, except in the geometric form of Euclid. Here the Greeks led the way, and the equation of the third degree appeared first in the famous problem of the duplication of the cube. This leads to $x^3 = 2a^3$ where x is the edge of a cube having a volume double that of the cube whose edge is a .

31

Algebra is a fascinating study with a notation and rules all its own. It is easier than geometry, because reasoning on the original problem is not required at every step, this reasoning being done automatically, as it were, by the operations on the symbols. It is a kind of generalized arithmetic whose rules and operations throw much light on the process of common arithmetic. The first important points to be learned are the properties of the signs $+$ and $-$ in multiplication, and when the student thoroughly understands that $-a$ multiplied by $-b$ gives $+ab$, he has entered upon a new field of intellectual pleasure. All young people ought to know something of the elements of algebra.

ALGEBRAIC AMUSEMENTS

32

Methods much in vogue a hundred years ago for solving simple equations were those called Single Position and

Double Position. The former may be illustrated by the following problem: What number is it of which its double, its half, and its third are equal to 34? Assume the number to be 48; then its double is 96, its half is 24, and its third is 16, and the sum of these is 136. Then state the proportion, the required number is to the assumed number as the given result is to the computed result, or $x : 48 :: 34 : 136$ from which the required number is 12. Why this proportion gives the correct result was rarely explained, so that the method was one of memory rather than of reasoning.

33

One of the problems put to Leonardo de Pisa at the contest before the Emperor Fredrick II in 1225 (No. 5) was as follows, the letters u, x, y, z , being introduced to simplify the enunciation: Three men A, B, C , possess a sum of money u , their shares being in the ratio $3 : 2 : 1$. A takes away x , keeps half of it, and deposits the remainder with D ; B takes away y , keeps two-thirds of it, and deposits the remainder with D ; C takes away all that is left, namely z , keeps five-sixths of it, and deposits the remainder with D . The deposits with D are found to belong to A, B, C , in the proportions $3 : 2 : 1$. Find u, x, y, z . Leonardo showed that the problem has many solutions, one of these being $u = 47, x = 33, y = 13, z = 1$.

34

A positive quantity a^2 has two square roots $+a$ and $-a$. But a negative quantity $-a^2$ has no square roots except the imaginary ones $+a\sqrt{-1}$ and $-a\sqrt{-1}$. The imaginary quantity $\sqrt{-1}$ first came to light through algebraic equations. Thus the equation $x^2 - 6x + 11 = 0$ has the two

roots $x = 3 + \sqrt{-2}$ and $x = 3 - \sqrt{-2}$, each of which satisfies the equation. Such roots were at first called impossible values, but later it was found that they have a good geometric representation, which will be spoken of later.

35

A quantity has three cube roots. For example, the three cube roots of 8 are $+2$, $-1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$, as may be easily verified. Likewise a quantity has four fourth roots, five fifth roots, and so on.

36

The symbol $\circ/\circ$ indicates an indeterminate quantity whose value can be found by consideration of the quantities which give rise to it. What is the value of $(a^3 - b^3)/(a - b)$ when a and b are each equal to unity? Most beginners in algebra will reply $\circ$. But, performing the division indicated, it will be found that $(a^3 - b^3)/(a - b)$ has the value $a^2 + ab + b^2$ which becomes 3 when $a = 1$ and $b = 1$. Hence for this case the value of $\circ/\circ$ is 3.

37

The equation $a^x + b^x = c^x$, in which a, b, c are integers, cannot be satisfied except when $x = 2$. For this value of x there are numerous solutions, as may be seen in the next chapter. Fermat's "last theorem" states that integral values for a, b, c cannot be found when x is greater than 2. The equation $a^x + b^x + c^x = d^x$ can, however, be satisfied in integers when $x = 3$, and $3^3 + 4^3 + 5^3 = 6^3$ is one solution.

38

Ask a person to think of a number while you call it x , then ask him to multiply it by 2 and add 6 to the product; this gives you $2x + 6$. Then ask him to square the number he thought of and add it to this, which gives you $x^2 + 2x + 6$. Next he is to subtract 5 from this, which gives you $x^2 + 2x + 1$. He is now asked to take the square root of this, and you have $x + 1$. If now he gives you this result, you have only to subtract 1 from it to know the number he thought of at first. Endless exercises like this can be made by the use of a little algebra.

39

There is a rule for determining whether or not a given number is a prime, but it gives rise to such large numbers that its use is generally impracticable. Let $n!$ denote the product of the first n integral numbers or $n! = 1 \times 2 \times 3 \times \dots \times n$. Then if $n! + 1$ is exactly divisible by $n + 1$ the number $n + 1$ is a prime. For example, to find if 7 is a prime, let $n + 1 = 7$ or $n = 6$, then $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, and $720 + 1 = 721$ which is exactly divisible by 7; hence 7 is a prime.

40

The following interesting problem is said to be due to Sir Isaac Newton: Three cows eat in two weeks all the grass on two acres of land, together with all the grass which grows there in the two weeks. Two cows eat in four weeks all the grass on two acres of land, together with all the grass which grows there in the four weeks. How many cows, then, will eat in six weeks all the grass on six acres of land together with all the grass which grows there in the six weeks? In

this problem it is, of course, understood that the quantity of grass on each acre is the same when the cows begin to graze, and also that the rate of growth is uniform during the time of grazing. Let x be the quantity of grass which grows on one acre in one week. Then from the statements of the problem it is not difficult to show that x is the same as the quantity of grass which was on one-fourth of an acre when the grazing began. The reader can now easily complete the solution without algebra and show that the answer to the problem is five cows.

ALGEBRAIC FALLACIES

41

A fallacy is improper reasoning which leads to an absurd result. Some of the fallacies are very puzzling to a beginner. The improper use of the quantity $0/0$ lies at the basis of an extensive series of fallacies. For example, take the equation $15x + 12 = 6x + 30$. This may be written $15x - 30 = 6x - 12$ or $5(3x - 6) = 2(3x - 6)$; dividing by $3x - 6$ gives the absurd result $5 = 2$. Here the true solution is $3x - 6 = 0$ or $x = 2$. Hence the last equation is really $5 \times 0 = 2 \times 0$ or $5 = 2 \times 0/0$. The value of $0/0$ is not 1 as the previous division has incorrectly assumed, but it is $5/2$, so that the true result of the division is the correct conclusion $5 = 5$. The following fallacies rest upon this improper use of $0/0$ and it is left to the reader to detect the place where the incorrect reasoning begins.

42

To prove that the numbers 1 and 3 are equal. Let $a = b$, then $ab^2 = a^3$; subtracting b^3 from both members gives $ab^2 - b^3 = a^3 - b^3$; dividing by $a - b$ gives $b^2 = a^2 +$

$ab + b^2$; in this make $a = 1$ and $b = 1$, and there results $1 = 3$.

43

Any number a is equal to a smaller number b . Let c be their difference so that $a = b + c$; if this equation be multiplied by $a - b$, there is found

$$a^2 - ab = ab - b^2 + ac - bc$$

or

$$a^2 - ab - ac = ab - b^2 - bc.$$

Dividing both members of the last equation by $a - b - c$, there results $a = b$, which proves the proposition enunciated.

44

To prove that $-1 = +1$. Let $x^2 - 1 = 0$; divide by $x + 1$ and there is found $x - 1 = 0$ or $x = +1$; divide by $x - 1$ and then $x + 1 = 0$ or $x = -1$. Therefore $-1 = +1$.

45

To solve the equation $5 + \frac{9x - 55}{7 - x} = \frac{4x - 20}{15 - x}$. Reducing the first member so that $7 - x$ shall be the common denominator of both terms gives

$$\frac{4x - 20}{7 - x} = \frac{4x - 20}{15 - x} \quad \text{or} \quad \frac{1}{7 - x} = \frac{1}{15 - x},$$

from which results the theorem that $7 = 15$!

46

Another class of fallacies embraces those which neglect to consider that a quantity has two square roots of equal value except that one root is positive and the other nega-

tive. As an example take the true equation $16 - 48 = 64 - 96$; add 36 to each member giving $16 - 48 + 36 = 64 - 96 + 36$. Each member is now a perfect square or $(4 - 6)^2 = (8 - 6)^2$. Taking the square root of each side, gives $4 - 6 = 8 - 6$ or the absurd result $4 = 8$. The fallacy here lies in taking the wrong square root, the correct extraction for this case being $(4 - 6) = -(8 - 6)$ which gives the correct conclusion $-2 = -2$. The following fallacies are based upon the neglect to consider all the roots of a quantity.

Solve the equation $x + 2\sqrt{x} = 3$. Proceeding in the usual manner there are found $x = 1$ and $x = 9$. The first satisfies the equation, but the second does not. Will the reader explain?

47

To solve the equation $x - a = \sqrt{x^2 + a^2}$; squaring both sides and reducing gives $-2ax = 0$, whence $x = 0$. But this root does not satisfy the given equation and hence the solution cannot be correct. Where lies the fallacy?

Any two numbers are equal to each other. Let a and b be the two numbers and assume the equation $(x - a)^3 = (x - b)^3$. Taking the cube root of both members gives $x - a = x - b$, whence $a = b$. Perhaps the advanced student may be able to show that the equation $(x - a)^3 = (x - b)^3$ has the three roots $x = \frac{1}{2}(a + b) \pm \frac{1}{6}\sqrt{-3}(a - b)$ and $x = \frac{1}{2}(a + b)$.

To solve the equation $x - a = (x^2 - a\sqrt{x^2 + a^2})^{\frac{1}{2}}$. Square each member twice and there will be found $x = 4/3a$ and $x = 0$. The first root satisfies the equation, but the second does not. Why not? Here it may be shown that $x = \infty$ is the true value of the second root.

48

Absurd fallacies like the following make good amusement for evening parties of young people: (1) Given the equations 32 ounces = 2 pounds, and 8 ounces = $1/2$ pound; multiplying these equations, member by member, gives 256 ounces = 1 pound; where is the fallacy? (2) Each of the following statements is true; 1 cat has 4 legs, 0 cat has 2 legs; add these, member by member, and there is found 1 cat has 6 legs; where lies the fallacy?

Sometimes beginners make mistakes like the following: $\sqrt{2} + \sqrt{3} = \sqrt{6}$; $a^2 + a^3 = a^6$; $x^0 = 0$; $x^{12} \div x^2 = x^6$; $x^{\frac{3}{2}} + x^{\frac{2}{3}} = x$. Where is the error in each of these?

49

What is the sum of the infinite series $1 - 1 + 1 - 1 + 1 -$, etc.? If 1 be divided by $1 + x$ there is found:

$$1/(1+x) = 1 - x + x^2 - x^3 + x^4 - x^5 +, \text{etc.},$$

and, making $x = 1$, it appears that the sum of the given infinite series is $1/2$. But suppose we divide 1 by $1 + x + x^2$, giving

$$1/(1+x+x^2) = 1 - x + x^3 - x^4 + x^6 - x^7 +, \text{etc.},$$

and then make $x = 1$, which shows that $1/3$ is the sum of the given infinite series. Which value is correct, $1/2$ or $1/3$?

50

The theory of logarithms belongs in Algebra. Since $(-a)^2 = a^2$ it may be thought that $\log(-a)^2 = \log a^2$ or $2 \log(-a) = 2 \log a$, from which it appears that $\log(-1) = \log 1$, but this is not true on account of a concealed fallacy. Other interesting problems may also be stated:

(1) What is the value of x in the equation $(3/4)^{\log x} + (4/3)^{\log x} = 25/12$? (2) What is the value of x in $e^{ix} = 1$, where e is the number 2.7128 and i is $\sqrt{-1}$?

51

A woman came to town with a basket of eggs. To the first customer she sold half her eggs and half an egg. To the second customer she sold half of the remaining eggs and one-half an egg. To a third customer she sold half of the remaining eggs and one-half an egg. Then counting the eggs in her basket she found exactly three dozen. How many eggs had she at the start?

At first thought this problem is impossible, for the idea of a woman selling half an egg and then walking along with the other half seems absurd. But trying algebra, let x equal the original number, then at the first transaction she sold $\frac{1}{2}x + \frac{1}{2}$ and had left $\frac{1}{2}x - \frac{1}{2}$. Thus proceeding to the last sale we put the final expression equal to 36 and find $x = 295$. One-half of 295 is $147\frac{1}{2}$ and $\frac{1}{2}$ an egg added makes 148. Thus she sold the first customer 148 eggs and had 147 left; to the second she sold $73\frac{1}{2} + \frac{1}{2}$ or 74 and then had 73 left; to the third she sold $36\frac{1}{2} + \frac{1}{2}$ or 37 and had 36 eggs left. Hence no division of an egg was necessary in making the three sales.

52

The following are problems proposed by the ancient Hindu mathematicians:

(1) The square root of half the number of bees in a swarm has flown out upon a jessamine bush, $8/9$ of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-flower into which he was

allured in the night by its sweet odor, but is now imprisoned in it. Tell me the number of the bees.

(2) A sixteen-year old girl slave costs 32 Uishkas, what costs one twenty years old by inverse proportion, the value of living creatures being regulated by their age, the older being the cheaper.

(3) Beautiful maiden with beaming eyes, tell me, as thou understandest the right method of inversion, what is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, by extraction of square root, addition of 8, and division of 10 gives the number 2?

(4) Of a flock of ruddy geese, ten times the square root of the number departed from the Manasa lake on the appearance of a cloud, an eighth part went to the forest of Sthala-padminis, and three couples were engaged in sport on the waters abounding with delicate flowers of the lotus. Tell me quickly, dear girl, the number of the flock.

53

It is said that the age of Diophantus when he died is known from the following problem: Diophantus was a child for one-sixth of his life, a youth for one-twelfth, and a bachelor for one-seventh; five years after his marriage a son was born who lived one-half as long as his father and who died four years before his father. When and where this problem was first propounded, I know not, but Cajori in his *History of Elementary Mathematics* says that it was an epitaph. Compute, young man, the age of Diophantus.

Plutarch relates the legend that the poet Homer died of vexation at being unable to solve a riddle propounded to

him by some young fishermen, in answer to his question as to how many they had caught. "As many as we caught," they said, "we left, as many as we did not catch, we carry."

54

THE CATTLE PROBLEM OF ARCHIMEDES

Condensed from *Popular Science Monthly*, November, 1905.

Lessing, librarian at Wolfenbüttel, discovered there about 1770 the manuscript of a Greek poem which enunciated a problem of great difficulty. The name of Archimedes appears in the title of the poem, it being said that he sent the problem in a letter to Eratosthenes to be investigated by the mathematicians of Alexandria. It may well be doubted, however, if Archimedes was the real author, since no mention of the problem has been found in ancient writings.

The following statement of the problem has been abridged from the German translations published by Nesselman in 1842 and by Krumbiegel in 1880:

Compute, O friend, the number of the cattle of the sun which once grazed upon the plains of Sicily, divided according to color into four herds, one milk white, one black, one dappled, and one yellow. The number of the bulls is greater than the number of the cows and the relations between them are as follows:

White bulls	= $(1/2 + 1/3)$ black bulls + yellow bulls.
Black bulls	= $(1/4 + 1/5)$ dappled bulls + yellow bulls.
Dappled bulls	= $(1/6 + 1/7)$ white bulls + yellow bulls.
White cows	= $(1/3 + 1/4)$ black herd.
Black cows	= $(1/4 + 1/5)$ dappled herd.
Dappled cows	= $(1/5 + 1/6)$ yellow herd.
Yellow cows	= $(1/6 + 1/7)$ white herd.

If thou canst give, O friend, the number of each kind of bulls and cows, thou art no novice in numbers, yet cannot be regarded as of high skill. Consider, however, the following additional relations between the bulls of the sun:

White bulls + black bulls = a square number.

Dappled bulls + yellow bulls = a triangular number.

If thou hast computed these also, O friend, and found the total number of cattle, then exult as a conqueror, for thou hast proved thyself most skilled in numbers.

It is seen that the exercise as stated involves two problems, the first to find integral numbers that satisfy the first seven conditions, and the second to find integral numbers that satisfy all nine conditions. The Wolfenbüttel manuscript has an Appendix giving 4 031 126 560 as the total number of cattle, but this satisfies only the first seven conditions. For this first problem let W , B , D , Y , represent the number of white, black, dappled, and yellow bulls, and let w , b , d , y , represent the number of white, black, dappled, and yellow cows. Then follow the seven equations:

$$W = \frac{5}{6} B + Y \quad (1) \qquad b = \frac{9}{20} (D + d) \quad (5)$$

$$B = \frac{9}{20} D + Y \quad (2) \qquad d = \frac{11}{30} (Y + y) \quad (6)$$

$$D = \frac{13}{42} W + Y \quad (3) \qquad y = \frac{13}{42} (W + w) \quad (7)$$

$$w = \frac{7}{12} (B + b) \quad (4)$$

and these contain eight unknown quantities. The problem, hence, is of the kind called indeterminate, for many sets of numbers may be found to satisfy the seven equations. That set having the smallest numbers is the one required, for any other set may be found by multiplying these by the same integer. If B and W are eliminated from equations (1), (2), (3) there will be found $891 D = 1580 Y$, and hence $D = 1580$ and $Y = 891$ are the smallest integral numbers satisfying it; from these are found $B = 1602$ and $W = 2226$. These numbers are now multiplied by an integer m and substituted in equations (4) to (7); then proceeding with the elimination, it is found that 4657 is the least value

of m that will make the results integral numbers. Accordingly,

$$B = 7\,460\,514$$

$$b = 4\,893\,246$$

$$W = 10\,366\,482$$

$$w = 7\,206\,360$$

$$D = 7\,358\,060$$

$$d = 3\,515\,820$$

$$Y = 4\,149\,387$$

$$y = 5\,439\,213$$

are the least numbers satisfying the conditions of the first problem. The total number of cattle is 50 389 082, not too many to graze upon the island of Sicily, the area of which is about 7 000 000 acres.

The second or complete problem includes the determination of numbers which not only satisfy equations (1) to (7), but also

$$W + B = \text{a square number}, \quad (8)$$

$$D + Y = \text{a triangular number}, \quad (9)$$

and this is to be done by finding an integer N to multiply into each of the results of the first problem, or

$$17\,826\,996\,N = \text{a square number},$$

$$11\,507\,447\,N = \text{a triangular number}.$$

A number N that will satisfy one of these conditions can be found without difficulty but to determine N so that both conditions will be satisfied is a task involving an enormous amount of time and labor. In fact, this required number N has never been completely computed.

A solution which satisfies (8) as well as (1) to (7) is easily made. Since $W + B$ is 17 826 966 N or $4 \times 4\,456\,749\,N$, and since 4 456 749 contains no number that is a perfect square, it is plain that N must be 4 456 749. Accordingly, each of the numbers found in the first solution must be multiplied by this value of N in order to satisfy (1) to (8) inclusive; the number $W + B$ is, then 79 450 446 596 004

which is a perfect square, but $D + Y$ is 51 285 802 909 803 which is not a triangular number.

It is now time to explain what is meant by a triangular number. The number ten is triangular because ten dots can be arranged in rows in the form of a triangle, the first row having one dot, the second two, the third three, and the fourth four dots. The next higher triangular number is 15 and the next 21, and in general $\frac{1}{2}n(n+1)$ is a triangular number whenever n is an integer, n being the number of rows in the triangle. The number 51 285 802 909 803 is shown not to be a triangular number by equating it to $\frac{1}{2}n(n+1)$ and computing n from the quadratic equation thus formed when it is found that n is not an integer.

Now since 51 285 802 909 803 is the number of yellow and dappled bulls which results from a solution which satisfies equations (1) to (8) inclusive, it is plain that the ninth condition may be expressed by

$$51\,285\,802\,909\,803\,x^2 = \frac{1}{2}n(n+1),$$

in which x and n are to be integers. When x^2 has been found each of the numbers of the first solution is to be multiplied by $4\,456\,749\,x^2$ in order to give the number of bulls and cows in each herd which satisfy the nine imposed conditions.

These numbers were readily seen to be so great that the island of Sicily could not contain all the cattle the problem seems to demand. This requirement, however, was understood to be only figurative, and mathematicians agreed that the numbers might be found altho no useful purpose would be attained by computing them. Thus the question rested until 1860 when Amthor demonstrated that 206 545 figures are necessary to express the total number of

cattle and that 766×10^{206542} gives their approximate number. This is an enormous number, and it is not difficult to show that a sphere having the diameter of the milky way, across which light takes ten thousand years to travel, could contain only a part of this great number of animals, even if the size of each is that of the smallest bacterium.

It would be thought that, after this investigation of Amthor, the cattle problem would have been finally dropped but such was not the case. The way to solve it was well understood from the theory of indeterminate analysis. Let the preceding equation be multiplied by 8, unity be added to each member and let $2n + 1$ be called y ; then it becomes

$$y^2 - 410\,286\,423\,278\,424x^2 = 1$$

which is of the form $y^2 - Ax^2 = 1$, and it is known that when A is an integer there can always be found integral values of x and y which satisfy the equation. The method of solution cannot well be explained here, but it was devised many years ago by Pell and Fermat and is well known to those skilled in higher mathematics. For example, take the simple case where $A = 19$, or $y^2 - 19x^2 = 1$, then the smallest integral values of y and x are 170 and 39.

In 1889 A. H. Bell, a surveyor and civil engineer of Hillsboro, Illinois, began the work of solution. He formed the Hillsboro Mathematical Club, consisting of Edmund Fish, George H. Richards, and himself, and nearly four years were spent on the work. They computed thirty of the left-hand figures and twelve of the right-hand figures of the value of x without finding the intermediate ones. This value is $x = 34\,555\,906\,354\,559\,370 \dots 252\,058\,980\,100$ in which the dots indicate fifteen computed figures, which it is unnecessary to give here, and 206 487 uncomputed

ones; the total number of figures in this number is 206 531. The final step is to multiply each of the numbers of the first solution by 4 456 749 and by this value of x^2 , thus giving:

White bulls	=	1 596 510		341 800
Black bulls	=	1 148 971		178 600
Dappled bulls	=	1 133 192		894 000
Yellow bulls	=	639 034		026 300
White cows	=	1 109 829		564 000
Black cows	=	735 594		645 400
Dappled cows	=	541 460		318 000
Yellow cows	=	837 676		113 700
Total cattle	=	7 760 271		081 800

in which each line of dots represents 206 532 figures, the total number of figures in each line being either 206 545 or 206 544. In each of these lines there are omitted twenty-four figures at the left end and six at the right end which were computed by the Hillsboro Mathematical Club.

This solution is published in the American Mathematical Monthly for May, 1895, where Bell remarks that each of these enormous numbers is "one-half mile long." A clearer idea of its length may be obtained by considering the space required to print it. Each page of this volume contains 32 lines and in each line about 50 figures may be printed, so that one page could contain about 1750 figures. To print a number of 206 245 figures would require 129 pages, and to print the nine numbers indicated above a volume of over 1000 pages would be needed.

It is known that Archimedes speculated regarding large numbers, for his book *Arenarius* is devoted to showing that a number may be written that will express the number of grains of sand in a sphere of the size of the earth. It cannot be proved that Archimedes was the author of the cattle problem, but as Amthor remarks, the enormous

numbers in its solution render it worthy of his genius and proper to bear his name. Its closing challenge still remains open, for the complete solution has not yet been made; and the investigations of Bell show that this would require the work of a thousand men for a thousand years. The little prairie town of Hillsboro, may, however, well exult as a conqueror, for its mathematical club has made the most complete of all solutions of the cattle problem and has proved itself to be highly skilled in numbers.

55

MAGIC SQUARES

Fig. 4 shows the well-known magic square containing the nine digits, the sum of each row, column, and diagonal being 15. These numbers may be arranged in other ways, for instance, by taking the left-hand column as the top row, the middle column as the middle row, and the right-hand column as the lowest row. Altogether there are eight different arrangements for this simplest of all magic squares.

4	9	2
3	5	7
8	1	6

Fig. 4

1	14	15	4
12	7	6	9
8	11	10	5
13	2	3	16

Fig. 5

16	2	11	5
3	13	8	10
6	12	1	15
9	7	14	4

Fig. 6

A magic square of sixteen numbers is shown in Fig. 5, the sum of each row, column, and diagonal being 34. Here also there are many other arrangements. Fig. 6 shows another magic square which has the same properties and there are others which will be explained later.

A true magic square should have 1 for its smallest number and contain all the natural numbers up to n^2 , where n is the

number of cells in one row or column of the square. The sum of the first n^2 natural numbers is $S = \frac{1}{2} n^2 (n^2 + 1)$, and hence the sum of the numbers in one row, column, or diagonal is $N = \frac{1}{2} n (n^2 + 1)$. Accordingly, for magic squares of 9, 16, 25, 36, 49, 64 cells the fundamental data are:

Cells in one row	$n =$	3	4	5	6	7	8
Sum of all numbers	$S =$	45	136	325	666	1225	2080
Sum of one row	$N =$	15	34	65	111	175	260

A magic square with 25 cells has the sum of the numbers in each row, column, or diagonal equal to 65. To form such a square write the numbers 1, 2, 3, 4, 5 in each row of the square in Fig. 7 so that the mean number 3 always comes

3	4	5	1	2
2	3	4	5	1
1	2	3	4	5
5	1	2	3	4
4	5	1	2	3

Fig. 7

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

Fig. 8

18	24	5	6	12
22	3	9	15	16
1	7	13	19	25
10	11	17	23	4
14	20	21	2	8

Fig. 9

in one of the diagonals, then write the numbers 0, 5, 10, 15, 20 in the rows of Fig. 8 so that the mean number 10 always comes in the opposite diagonal. For Fig. 7 the sum of each row, column, and diagonal is 15 and for Fig. 8 it is 50, the total being 65. Then add the numbers in corresponding cells and put the results in Fig. 9 thus forming a magic square of 25 cells where the sum of each row, column, and diagonal is 65.

Another magic square of 25 cells may be formed by taking the first five natural numbers in the order 4, 1, 3, 2, 5 and placing them in each row of a square in this order, 3 always coming in a diagonal cell; then in another square writing 15, 0, 10, 5, 20 so that 10 comes in each cell of the

opposite diagonal; and finally adding the numbers in corresponding cells. Other squares may be formed by writing the first five natural numbers in different orders, keeping always 3 in the middle, and arranging the auxiliary numbers correspondingly. Altogether twelve magic squares may be formed in this way, and from each of these still others may be formed by interchanging rows and columns.

Any magic square with an odd number of cells n in one row, can be formed in a similar way, by writing the first n natural numbers in each row so that $\frac{1}{2}(n+1)$ comes in a diagonal cell, then writing the numbers $0, n, 2n, 3n, \dots (n-1)n$, so that $\frac{1}{2}(n-1)n$ comes in the cells of the opposite diagonal, and finally adding the numbers in corresponding cells. Thus for $n=7$, the first set of numbers might be 5, 3, 1, 4, 6, 2, 7, when the second set must be 28, 14, 0, 21, 35, 7, 42; here 4 must be written along one diagonal and 21 along the other diagonal.

The formation of magic squares having an even number of cells is not so easy and it seems that a general rule has not been given. For 16 cells, however, the following rule is applicable. Write in the upper and lower rows of a square the numbers 1, 3, 2, 4; then in the two middle rows write them in the reverse order. Again in the top row put the numbers 0, 12, 12, 0, and in the lowest row write 12, 0, 0, 12; in the upper middle row put 8, 4, 4, 8 and in the one below it 4, 8, 8, 4. The addition of these numbers will give a magic square of 16 numbers which will be slightly different from those shown in Figs. 5 and 6.

A magic square of 64 cells may be seen on page 163 of Ball's *Mathematical Recreations and Essays* (London, 1911), which possesses the wonderful property that if each number be replaced by its square the resulting square is

also magic, the sum of the numbers in each line being 11180.

We shall now briefly mention the magic squares sometimes called "diabolic," or more commonly "Nasik," this being the name of the town in India where A. H. Frost invented them. Fig. 6 shows a Nasik magic square of 16 cells where the sum of each row, column, and diagonal is 34. The sum of the numbers in each broken diagonal is also 34, a broken diagonal being one which is partly on one side of the main diagonal and partly on the other side; thus the numbers 11, 13, 6, and the number 4 constitute a broken diagonal, as likewise do the numbers 2, 3, and 15, 14. In this magic square the sum of the numbers in any small square formed by four adjacent cells is also equal to 34. Truly, this is a marvellous arrangement of the first sixteen natural numbers.

Benjamin Franklin, the famous philosopher and diplomat, amused himself with magic squares. At page 251 of Volume 3 of his *Collected Works* (Philadelphia, 1808) may be seen a square having eight cells on a side, or 64 cells in all, in which the sum of each row and column is 260, while the sum of the numbers in any four adjacent cells is 130. This, however, is not a true magic square, as the sum of the numbers is 292 for one diagonal and 228 for the other. Franklin also devised a square of 2056 cells which is called the "magic square of squares," and a magic circle having many curious properties.

Squares having the sum of the numbers in each line greater than $\frac{1}{2}n(n^2 + 1)$ may be formed by adding an integer I to each number so that the sum of all the numbers in each line is $N = \frac{1}{2}n(n^2 + 1) + nI$. When N is given, values of n may sometimes be found which satisfy

this equation. Thus for $N = 1000$ and $n = 4$ the value of I is integral, namely, 187, so that if 187 be added to each of the numbers in Fig. 6, the resulting numbers have the property that the sum of each row, column, and diagonal is 1000. The greatest and least numbers in such a square may be found from the expression $N/n \pm \frac{1}{2}(n^2 - 1)$; thus for $N = 1000$ and $n = 5$ they are 212 and 188.

Among other curious squares are those which are filled with the first n^2 natural numbers by the knight's move in chess, each square being occupied only once by the knight. Leonard Euler, the great mathematician, amused himself with such squares and two which he constructed for squares of five and seven sides may be seen in the *Encyclopedia Britannica*; these squares, however, are not magic, although they have certain curious properties. It is well known that this problem may be solved on the common chess board in many different ways; and one of these gives a square which is magic.

CHAPTER III

GEOMETRY

56



THE QUEEN of mathematics is the ancient geometry as exemplified by Euclid. Elegant, chaste, and beautiful is its logic, wonderful are its conclusions. It originated in Egypt and came to its development at the great university of Alexandria where Euclid was the founder of its mathematical school. The Greek words $\gamma\eta$ and $\mu\epsilon\tau\rho\omicron\nu$, which form the name of the science, mean land and measure, respectively, so that geometry was originally the measurement of land. In Egypt where the annual inundations of the Nile easily obliterate the boundaries of parcels of land, perhaps the rules of geometry received their first practical application.

57

Euclid lived about 300 B. C. Tradition says that he was mild and unpretending in manner and kind to all genuine students of mathematics. On one occasion a student complained that geometry brought no profit, whereupon Euclid directed that three oboles be given him. To King Ptolemy, who asked if there was not an easier way to understand geometry than through study of the Elements, Euclid gave the reply "there is no royal road to geometry."

58

The Elements of Euclid is the title given to that presentation of plane geometry which Euclid prepared about 300

B. C. Written in Greek, it later was translated into Arabic, then into Latin. Translations have appeared in all European languages which have been annotated by many different editors. More than a thousand editions have appeared since the invention of printing about 1480; in fact, no book except the Bible has passed through so many editions as the Elements of Euclid.

The first six books of Euclid were generally used in colleges and schools in England and America until about 1860. The logic of the presentation is generally perfect, and the treatise gives important facts of plane geometry. It can be successfully used today, for it is certain that it is much better than some texts now on the market and equally as good as many of them.

GEOMETRIC AMUSEMENTS

59

The second proposition of the first book of Euclid affords amusement to some beginners because it appears to them that a much simpler method might have been used. The first proposition is to construct an equilateral triangle upon a straight line of given length. The second is "to draw from a given point a line equal to a given straight line"; to do this Euclid lets A be the given point and BC the given straight line; then joining A and B he constructs upon AB the equilateral triangle ABD by the method of the first proposition. From B as a center with a radius BC the circle CC' is described and DB is produced until it meets the circle in E . Then from D as a center with the radius DE a

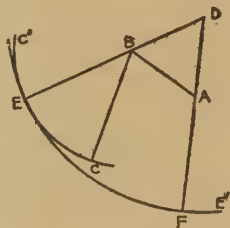


Fig. 10

second circle EE' is described and DA is produced until it meets the circle in F . Accordingly, from the given point A the straight line AF has been drawn equal to the given straight line BC . The reader can easily supply the steps of the demonstration, but can he state why Euclid did not at once describe from the center A a circle with the radius BC ?

60



Fig. 11

The fifth proposition of the first book of Euclid has always been known as the “pons asinorum,” and it has been generally implied that the boy who failed to understand it was an ass. The word “pons” or “bridge” perhaps originated from the figure which roughly resembles the rude bridge truss used to span narrow streams.

61

The most important proposition in the first book of Euclid's *Elements of Geometry* is the forty-seventh, namely: The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. This truth was known to Hindoos and Egyptians long before the time of Euclid, but Pythagoras, who lived 550 B. C., gave a formal demonstration which has caused his name to be frequently applied to the theorem.

62

There are many right-angled triangles having the three sides expressed by integral numbers. The simplest one has 3, 4, 5 for its sides and this was known to the Egyptians who

used it many centuries before the Christian era in constructing the pyramid of Cephren at Gizeh. If each of these sides be doubled we have 6, 8, 10 for the sides of a right-angled triangle which has probably been known to surveyors in all ages and which is now constantly used by them in laying off a right angle with the chain. The following are all the right-angled triangles with sides expressed in integers for which the shortest side does not exceed 15:

$3^2 + 4^2 = 5^2$	$11^2 + 60^2 = 61^2$
$5^2 + 12^2 = 13^2$	$12^2 + 16^2 = 20^2$
$6^2 + 8^2 = 10^2$	$12^2 + 35^2 = 37^2$
$7^2 + 24^2 = 25^2$	$13^2 + 84^2 = 85^2$
$8^2 + 15^2 = 17^2$	$14^2 + 48^2 = 50^2$
$9^2 + 12^2 = 15^2$	$15^2 + 20^2 = 25^2$
$9^2 + 40^2 = 41^2$	$15^2 + 36^2 = 39^2$
$10^2 + 24^2 = 26^2$	

63

Let m and n be any two integers; then if $2mn$ be taken for one of the legs of a right-angled triangle, the other leg is $m^2 - n^2$ and the hypotenuse is $m^2 + n^2$. Thus let $m = 9$ and $n = 4$, then the three sides are 72, 65, 97. Let the reader use this rule to determine three right-angled triangles each having 48 for one of its legs.

64

The following are some of the other right-angled triangles, with integral sides, that have the same area: first triangle, 24, 70, 74; second triangle, 40, 42, 58; third triangle, 15, 112, 113.

65

A castle wall there was, whose height was found
To be just fifty feet from top to ground;

Against the wall a ladder stood upright,
 Of the same length the castle was in height.
 A waggish fellow did the ladder slide,
 (The bottom of it) five feet from the side.
 Now I would know how far the top did fall
 By pulling out the ladder from the wall?

66

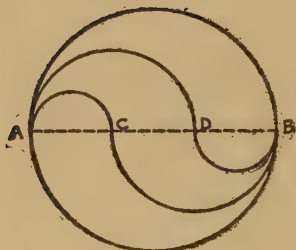


Fig. 12

To divide a circle into three equal parts. One method is as follows: Divide a diameter AB into three equal parts AC , CD , DB ; on AC and CB describe the semi-circles shown, also on AD and DB describe semi-circles; then is the area of the given circle divided into three equal parts.

67

When is the sum of the squares of two successive integers a perfect square? This is answered by Osborne in *American Mathematical Monthly* for May, 1914. The two smallest are $0^2 + 1^2 = 1^2$ and $3^2 + 4^2 = 5^2$. Then come $20^2 + 21^2 = 29^2$ and $119^2 + 120^2 = 169^2$. Five larger sets are also found the largest of which is $803\,760^2 + 803\,761^2 = 1\,136\,689^2$.

68

To divide a circle into n equal parts an approximate solution is the following: Let AB be the diameter of the given circle and CD a diameter perpendicular to AB . Divide AB into n equal parts. On the diameter DC produced lay off CE equal to one-third of the diameter. From E draw a straight line through the second point of division

on AB and produce it to meet the circumference in F . Then the arc AF is the n th part, very closely, of the circumference.

69

The trisection of the angle is one of the famous problems of antiquity. It can be solved in many ways, but not by plane geometry, for Euclid allowed no instruments but a straight ruler and the compasses. However, if it be permitted to mark or graduate the ruler, the problem can be solved in the following manner, as was first shown by Archimedes. Let BAC be the angle to be trisected and from A as a center describe a semi-circle with the compasses.

Produce the radius BA toward the right. From one end of the ruler lay off on it a distance equal to the radius AC and mark the point thus found. Place the ruler

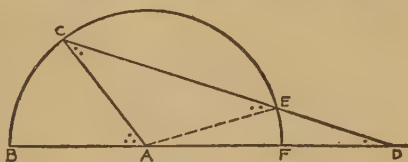


Fig. 13

so that one edge coincides with C while the end moves along the produced line BD . When the mark on the ruler coincides with the semi-circle put there a point E , then the arc EF is one-third of the arc BC and the angle EAF is one-third of the given angle BAC . The reader can easily supply the demonstration by considering the dots that have been placed on the figure.

70

THE VALUE OF π

Archimedes deduced, about 220 B.C., a rule for the quadrature of the circle, proving that its area is equal to

the square of its radius multiplied by a number which lies between $3\frac{10}{71}$ and $3\frac{10}{70}$. Or, $\text{area} = \pi r^2$, where r is the radius and π is the number whose approximate value is $3\frac{1}{7}$. This number π is also the ratio of the circumference of the circle to its diameter and it turns up in connection with many problems not at all related to the circle.

71

The value of π to four decimal places is 3.1416. "Yes, I have a number," is a sentence in which the number of letters in each word corresponds to the integers in this value of π . The following, which appeared in the Scientific American of March 21, 1914, will enable its value to be remembered to 12 decimals:

See I have a rhyme assisting
My feeble brain its tasks sometimes resisting.

72

An early statement regarding the ratio of the circumference of a circle to its diameter is found in the Bible in connection with the description of Solomon's temple. The architect of this magnificent building was Hiram, a widow's son, whose father was a man of Tyre. In I Kings, vii, 23, and also in II Chronicles, iv, 2, we find the dimensions of a circular tank or pond which was designed by Hiram. "He made a molten sea, ten cubits from one brim to the other; it was round all about and its height was five cubits; and a line of thirty cubits did compass it about." It should not be inferred from this description, however, that the value $\pi = 3$ was used in computations by this distinguished architect. The date of the construction of Solomon's temple was about 1007 B. C.

73

The early Romans are said to have used $3\frac{1}{8}$ for the value of π but Frontinus, in 97 A. D., used $3\frac{1}{7}$, as is seen from the list of circumferences and diameters of water pipes which is mentioned in No. 3. He also used this value in computing areas, as appears from his statement: "The square digit is greater than the round digit by three-fourteenths of itself; the round digit is smaller than the square digit by three-elevenths."

The value of π was computed by William Shanks in 1873 to 707 decimal places, surely a great waste of labor, for the most refined computation requires only seven or eight decimals, and in all usual work 3.1416 is close enough. It has been proved that the number π is incommensurable, that is, the number of its decimals is infinite.

THE PYRAMIDS OF EGYPT

74

It has been claimed that the great pyramid at Gizeh in Egypt was intended to be so built that the length of the four sides of the base should be the circumference of a circle whose radius was the vertical height. Petrie's measurements of 1882 give 9068.8 inches for the length of one side of the base and 5776.0 inches for the height of the pyramid when its sides met at an apex. Now $9068.8/5776.0 = 1.5703$, whereas $\frac{1}{2}\pi$ is 1.5708. Probably they used $3\frac{1}{7}$ for the value of π ; in this case the ratio of one of the sides of the base to the height would have been $11/7$ or 1.5714. Petrie's measures of the angle made by the sides of the pyramid with the horizontal gave the mean value $51^\circ 52'$; this corresponds to 1.5735 for the above ratio.

The pyramid of Cephren at Gizeh has its sides inclined to the horizontal at an angle of $53^{\circ} 10'$. This corresponds very closely to a slope of 4 on 3, so that the right-angled triangle having sides of 3, 4, 5 seems to have been used in its construction. Here the ratio of one side of the base to the height is $6/4 = 1.5$ instead of 1.57 as in the other pyramids at Gizeh. Undoubtedly mathematics and astrology controlled the design of these pyramids, altho their final purpose was for tombs for the kings. So mighty is the great pyramid at Gizeh and so solidly is it constructed that it will undoubtedly remain standing long after all other buildings now on the earth have disappeared.

75

Herodotus said that the area of an inclined face of the pyramid was equal to a square described upon its altitude. What value does this condition give for the angle which the plane of a face makes with the base?

76

The King's Chamber in the great pyramid is 10 cubits wide, 20 cubits long, and 11.18 cubits high. These figures result from Petrie's measurements made in English inches, 20.612 inches being taken for the length of the old Egyptian cubit. The height was hence made one-half of the floor diagonal, so that the three dimensions of the room are 10, 20, $\frac{1}{2}\sqrt{500}$ cubits, and the solid diagonal is 25 cubits in length. These numbers are proportional to 1, 2, $\frac{1}{2}\sqrt{5}$, 2.5. It can hardly be supposed that these dimensions were accidental; they were probably introduced into the design in accordance with some astrological superstition of a mathematical nature.

77

A magnitude or quantity is anything that can be measured. Can a solid angle, like that at the apex of a pyramid, be measured? No one ever spoke of a solid angle as being twice as large as another one. The only measure of a solid angle that has been proposed is the surface of a sphere described from its apex and included between its sides. The radius of the sphere being taken as unity, $\frac{1}{4} \pi$ would be the measure of the solid angle at one corner of a cube. What is the measure of the solid angle at the apex of a right cone whose altitude is equal to the radius of its base?

78

THE PRISMOIDAL FORMULA

A general method of finding the volume of any of the solids of common geometry is the Prismoidal Formula. Let A and B be the areas of the two parallel bases and C the area of a parallel section halfway between them; let h be the altitude between the bases A and B . Then the volume of the solid is $V = \frac{1}{6} h (A + 4C + B)$.

To apply this to a cone which has a base of radius r and the altitude h , the upper base A is 0, since it is at the apex of the cone, the lower base B is πr^2 , and C , the area of a section halfway between the two bases is $\frac{1}{4} \pi r^2$. Then the volume is $V = \frac{1}{6} h (0 + \pi r^2 + \pi r^2) = \frac{1}{3} \pi r^2 h = \frac{1}{3} Bh$.

To find the volume of a sphere draw two parallel planes tangent to it, giving the two bases $A = 0$ and $B = 0$; the area of a section halfway between them is πr^2 , where r is the radius of the sphere; also the altitude h is $2r$. Then volume $= \frac{1}{6} (2r) (0 + 4\pi r^2 + 0) = \frac{4}{3} \pi r^3$.

To find the volume of a masonry pier 16 feet high, the

top B being a rectangle 8×24 feet, and the lower base being a rectangle 12×30 feet inside. The areas of the bases are 192 and 360 square feet. The dimensions of a section C halfway between the bases are $\frac{1}{2}(8 + 12)$ or 10 feet and $\frac{1}{2}(24 + 30)$ or 27 feet, so that the area of C is 270 square feet. Then

$$\text{Volume} = \frac{1}{6} \times 16 (360 + 1080 + 192) = 4352 \text{ cu. ft.}$$

This is a problem which is difficult to solve by the methods of common geometry, for the sides of the pier when produced do not meet at a point, and hence the rule for a truncated pyramid does not apply.

The prismoidal formula gives volumes of the ellipsoid, paraboloid, and other solids generated by the revolution of curves of the second and third degree about an axis. It also applies to warped surfaces like the hyperbolic paraboloid when the areas A , B , C are known or can be found.

Let the student apply the prismoidal formula to find the volume of a segment of a sphere whose altitude is h and the radius of whose base is a . Here a little analytic geometry is perhaps necessary to find C in terms of a and the radius r of the sphere.

79

GEOMETRIC FALLACIES

To prove, geometrically, that 24 equals 25; draw a square on cardboard, 5 inches on a side, having an area of 25 square inches, as shown in Fig. 14; then cut the cardboard into four pieces as indicated by the three broken lines; these four pieces can then be arranged in the rectangular form shown in Fig. 15, where there are three inches on one side and eight on the other, giving twenty-four square inches in

- all. Hence it has been proved geometrically that 24 equals
 25. Where is the fallacy?

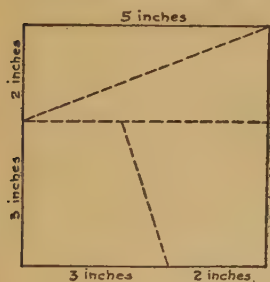


Fig. 14

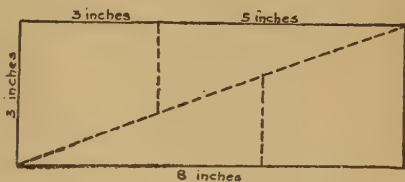


Fig. 15

80

To prove that a straight line can be divided into four parts so that the first point is to the third as the third is to the fifth. Let AE be the given straight line which is so divided that $AB : BC :: CD : DE$; then $AB/BC = CD/DE$; now cancelling B out of the first member of this equation and D out of the second, there results $A/C = C/E$, or $A : C :: C : E$, which proves the proposition enunciated.

81

The semicircumference of a circle is equal to its diameter. Let the diameter be divided into four equal parts and on each part let a semicircle be described. These four smaller semicircles are equal to the given semicircumference; for let d be the given diameter, then $\frac{1}{2} \pi d$ is the corresponding semicircumference; each of the equal parts of the diameter is $\frac{1}{4} d$ and the corresponding semicircumference is $\frac{1}{8} \pi d$; hence the sum of these four small semicircumferences is $4 (\frac{1}{8} \pi d)$ or $\frac{1}{2} \pi d$. Now divide each of the parts of the diameter into four equal parts and describe semicircles on

them and it is clear that the sum of the sixteen semicircumferences is equal to the large semicircumference originally given. Thus continue, and when the number of points of division is infinite the sum of all the infinitely small circumferences is equal to the large original one. But when this occurs all the small semicircumferences coincide with the diameter of the circle and their sum is hence equal to it. Accordingly, it has been proved that the semicircumference of any circle is equal to its diameter. Where lies the fallacy?

82

To prove that any obtuse angle is equal to a right angle. Let ABC be an obtuse angle and DCB a right angle; it is required to prove that these angles are equal. Make CD equal to BA , join AD , bisect it in E and draw EG perpendicular to AD . Also bisect BC in F and draw FG perpendicular to BC . The two perpendiculars meet in G . Draw

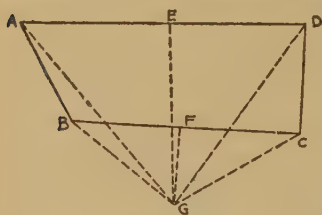


Fig. 16

GA , GD , GB , and GC ; then GA equals GD , and GB equals GC . Since also $CD = BA$, the sides of the triangle GBA are equal, each to each, to the sides of the triangle GCD . Hence these triangles are in every way equal; and the angle

opposite the side GA is equal to the angle opposite the side GD . Accordingly the angle ABG equals the angle DCG ; subtracting from these the equal angles CBG and BCG , the result is that the angle ABC is equal to the angle DCB . Therefore, it has been proved that the obtuse angle ABC is equal to the right angle DCB . The fallacy in this demonstration is not easy to detect, but nevertheless it is there.

83

The circumference of a small circle is equal to the circumference of a larger circle. Let a wheel roll along a horizontal plane until it has made one revolution; then the line AB is equal to its circumference. The small circle in



Fig. 17

the figure has also in the same time made one revolution, since it is drawn on the side of the wheel concentric with the larger circle. Hence the circumference of the small circle rolls out the line CD which is equal to AB . Therefore, the two circumferences are equal. Where is the fallacy?

84

Every triangle is isosceles, or two angles of any triangle are equal to each other. Let abc be any triangle. Bisect the angle a ; from the middle of bc draw a normal to bc ; the bisector and the normal meet at a point. From this point draw lines to b and c and normals to the sides ab and ac . The triangles C and D are then equal; also the triangles A and B are equal, whence $ad = ae$. Accordingly, the third pair of triangles E and F must be equal, whence $cd = be$. Hence $ad + cd = ae + eb$, or $ac = ab$. Thus it has been proved that any triangle is isosceles.

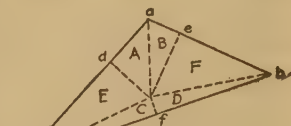


Fig. 18

The three following propositions are certainly interesting. Are they true or false? (1) Let AB be a straight line and C any point on it. On AC and BC as bases construct the isosceles triangles AbC and BaC so that the equal sides make angles of 30° with the bases. Also on AB construct the isosceles triangle AcB so that the equal sides make angles of 30° with AB . Draw ab , bc , ca as shown by the

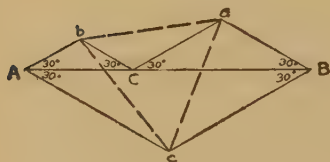


Fig. 19

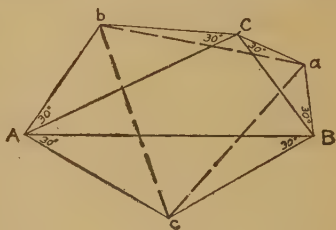


Fig. 20

broken lines. Then each of the angles of the triangle abc is 60° . (2) Let ABC be any triangle. On the sides as bases construct the isosceles triangles AcB , BcA , CbA , so that the equal sides of each make angles of 30° with its base. Draw ab , bc , ca , as shown by the broken lines. Then each of the angles of the triangle abc is 60° . (3) Describe a square on each of the sides of a right-angled triangle. At the centers of these squares put the points a , b , c , and join these points so as to form the triangle abc . Then each of the angles of this triangle is 60° .

The following remarkable fallacy appeared in the *Forum* of April, 1914: "A cube will readily present to the eye three dimensions: length, height, and breadth. Four

diagonal lines imagined from the corners of the cube will each be at right angles to the other three; hence we have four dimensions. We should find it difficult to construct anything along the lines of these four dimensions for the simple reason that the work would have to begin at the point where the lines intersect and progress outward through within the four lines. We might call these four lines expansion boundaries for if you would cause a cube to expand and maintain its symmetry or proportions, it would expand along these four lines. Any solid can therefore be considered a cross section of its greater self. The foregoing is the only practical demonstration that can be given of four dimensions."

87

ON THE AREA OF A CLOSED TANGLE

From Clifford's *Common Sense of the Exact Sciences*, Fifth Edition
(London, 1907), pages 135-137.

Hitherto we have supposed the areas we have talked about to be bounded by a single loop. It is easy, however,

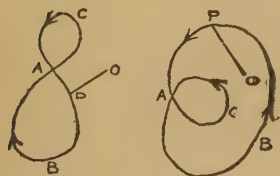


Fig. 21

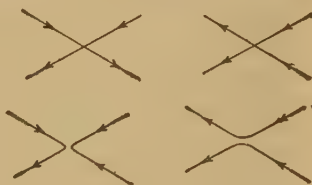


Fig. 22

to determine the area of a combination of loops. Thus, consider the figure of eight in Fig. 21 which has two loops; if we go around it continuously in the direction indicated

by the arrowheads, one of these loops will have a positive, the other a negative area, and therefore the total area will be their difference, or zero if they be equal. When a closed curve, like a figure of eight, cuts itself it is termed a *tangle*, and the points where it cuts itself are called *knots*. Thus a figure of eight is a tangle of one knot. In tracing out the area of a closed curve by means of a line drawn from a fixed point to a point moving around the curve, the area may vary according to the direction and the route by which we suppose the curve to be described. If, however, we suppose the curve to be sketched out by the moving point, then its area will be perfectly definite for that particular description of its perimeter.

We shall now show how the most complex tangle may be split up into simple loops and its whole area determined from the areas of its simple loops. We shall suppose arrowheads to denote the direction in which the perimeter is to be taken. Consider either of the accompanying figures (Fig. 21). The moving line OP will trace out exactly the same area if we suppose it not to cross the knot at A but first trace out the loop AC and then to trace out the loop AB in both these cases going around these two loops in the direction indicated by the arrowheads. We are thus able in all cases to convert one line cutting itself in a knot into two lines, each bounding a separate loop, which just touch at the point indicated by the former knot. This dissolution of knots may be suggested to the reader by leaving a vacant space where the boundaries of the loops really meet. The two knots in Fig. 22 are shown dissolved in this fashion.

The reader will now have no difficulty in separating the most complex tangle into simple loops. The positive or negative character of the areas of these loops will be suffi-

ciently indicated by the arrowheads on their perimeters. We append an example (Fig. 23).



Fig. 23

In this case the tangle reduces to a negative loop a and to a large positive loop b , within which are two other positive loops c and d , the former of which contains a fifth small positive loop e . The area of the entire tangle then equals $b + c + d + e - a$. The space marked s in the first figure will be seen from the second to be no part of the area of the tangle at all.

88

MAP COLORING

It has long been known that only four different colors are necessary in order to color the most complicated map of a country so that contiguous sides of districts shall not have the same color. About 1850 this fact was brought to the attention of mathematicians but, altho much discussion of it has been made, a rigorous proof that only four colors are necessary has not yet been made.

Fig. 24 shows a map of nine districts in which the four colors A, B, C, D are used for eight of the areas and there may seem no way to use one of these colors for the other district unless it adjoins upon the same color. However, by the very slight change shown in Fig. 25 the problem is

readily solved. Thus in all cases a way can be found to color the map by using only four different colors.

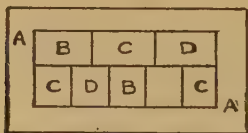


Fig. 24

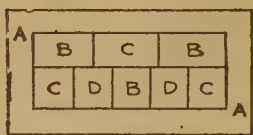


Fig. 25

The reason that five colors are not required seems to be that it is impossible to draw five areas so that a boundary of each shall be contiguous to the other four. Fig. 26 shows four areas each of which has its boundary contiguous

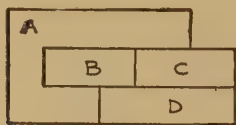


Fig. 26

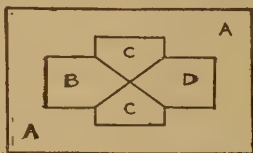


Fig. 27

with a boundary of the other three areas, but no way can be found to add a fifth area so that it may be contiguous to the other four. Four colors are sufficient for any map because no map has yet been drawn in which five areas are contiguous to four others. But no proof has yet been discovered that it is impossible to draw five such areas.

The word "contiguous" means that the areas border along a line, not at a point. Districts sometimes occur so that four or more of them meet at a point; for example, in Fig. 27 the two areas colored *C* meet at a point. Here more than four colors are needed if it is desired to have the areas on opposite sides of the point of junction different in shade.

CHAPTER IV

TRIGONOMETRY

89



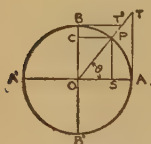
THE SOLUTION of triangles was the original object of Trigonometry, but it has been extended in modern times to include a vast realm of facts regarding functions of angles. The beginner in trigonometry is first introduced to the sine and cosine which are defined by a right-angled triangle in a manner essentially like the following: Let a be the hypotenuse and b and c the legs, and A , B , C the angles opposite to them, then the ratio b/a is called the sine of the angle B , and the ratio c/a is called the sine of C . Or as sometimes stated, the sine of an acute angle in a right-angled triangle is the ratio of the side opposite the angle to the hypotenuse. Thus $\sin B = b/a$ and $\sin C = c/a$.

Now it has been questioned by H. E. Licks whether this is the best way to define the sine for the beginner. The beginner is young and immature, to him the word "ratio" is more or less of an abstraction, and the fact that this ratio *is called* the sine does not appear to him significant. Why not state the definition something like this: The sine of an acute angle in a right-angled triangle is a number which multiplied by the hypotenuse gives the side opposite to the angle. This definition puts the matter in quite a different light for it gives the idea that the primary use of the sine is to solve a right-angled triangle, and it states the rule by

which one of the sides may be found when the sine of the opposite angle and the hypotenuse are known. How the sines are tabulated and used is a matter to be explained later.

90

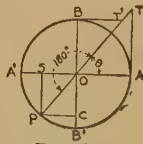
Fifty years ago a very different method of defining the trigonometric functions was in use. Let AOP in Fig. 28 be an angle less than 90° which is measured from the line AO around in a contrary direction to that of the hands of a watch. Let P be any point on the quadrant AB described with the radius OA or OP ; let BB' be a diameter normal to AA' . From the point P let perpendiculars PS and PC be



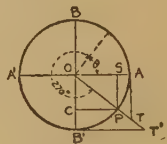
First Angle
Fig. 28



Second Angle
Fig. 29



Third Angle
Fig. 30



Fourth Angle
Fig. 31

drawn to the diameters AA' and BB' . Then, if the radius OA or OP be of units length, PS is the sine of the angle AOP and PC is its cosine. Also at A let a perpendicular be drawn to OA until it meets OP produced, then AT is the tangent of the angle AOP and OT is its secant; at B let a perpendicular to the diameter BB' be drawn until it meets OP produced, then BT' is the cotangent of the angle AOP and OT' is its cosecant.

Fig. 29 shows a similar representation for an angle AOP when P falls in the second quadrant, or for $90^\circ + \theta$. Fig. 30 shows the functions for $180^\circ + \theta$, and Fig. 31 for $270^\circ + \theta$, the value of θ being in each case less than 90° . In each figure PS and PC are the sine and cosine of AOP , while AT

and BT' are its tangent and cotangent and OT and OT' are its secant and cosecant.

Let the distances measured upward from AA' be taken as positive and those measured downward be negative. Let distances to the right of BB' be positive and to the left negative. Then the diagrams determine at once the signs of the trigonometric functions for the angles AOP in the different quadrants. Thus when P is in the first quadrant the angle AOP or θ is less than 90° , and its sine, cosine, and tangent are positive (Fig. 28). When P is in the second quadrant it lies between 90° and 180° , here the sine of AOP is positive, while its cosine and tangent are negative (Fig. 29). For the third quadrant AOP is between 180° and 270° , its sine and cosine are negative but its tangent is positive (Fig. 30). For the fourth quadrant AOP is between 270° and 360° , its cosine is positive and its sine and tangent are negative (Fig. 31). The cotangent is seen to be positive for the first and third quadrants and negative for the second and fourth. The secant and cosecant are taken as positive when measured from O toward P ; hence they are both positive in the first and fourth quadrants, and both negative in the second and third.

The diagrams clearly show how each of the functions varies as an angle increases from 0° to 360° . Take for example the tangent. When $\theta = 0$, $\tan \theta = 0$; as θ increases so does $\tan \theta$, when $\theta = 90^\circ$, $\tan \theta = +\infty$; then just as θ passes through 90° $\tan \theta$ becomes $-\infty$; as AOP increases towards 180° its tangent, though always negative, decreases numerically and becomes 0 when $AOP = 180^\circ$. In a similar manner the diagrams show that $\cos \theta$ is 1 when $\theta = 0^\circ$, decreases to 0 when $\theta = 90^\circ$, then becomes negative but increases numerically until its value

is -1 when AOP is 180° . Thus the variation of each of the functions can be clearly traced.

These diagrams can be readily carried in the mind of a student after they have been intelligently constructed by him. They show instantly such relations as $\sin(90^\circ + \theta) = \sin \theta$, $\cos(90^\circ + \theta) = -\cos \theta$, $\tan(180^\circ + \theta) = \tan \theta$, $\sin(270^\circ + \theta) = -\sin \theta$, etc., θ being considered for this purpose as less than 90° . They also show that $\cos(90^\circ - \theta) = \sin \theta$, $\cot(90^\circ - \theta) = \tan \theta$, etc. Even in these modern days they may be useful to some of the beginners in trigonometry who have been brought up in a severer logic. At any rate the reason why the word tangent is used for one of the functions will be clear to him.

91

There are few paradoxes or fallacies in trigonometry, but here is one which is worth preserving. To prove that the angles of a triangle are equal to each other. Let A, B, C be the angles of the triangle ABC , and a, b, c the sides opposite to them. Produce BA and make AD equal to c , then the angles ADB and ABD are each equal to $\frac{1}{2} A$. Produce CA and make AE equal to b , then the angles AEC and ACE are each equal to $\frac{1}{2} A$. Now from the triangle BCD , there can be written

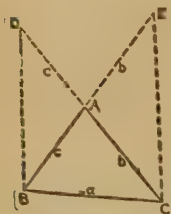


Fig. 52.

$$\frac{CD}{BC} = \frac{b + c}{a} = \frac{\sin(B + \frac{1}{2}A)}{\sin \frac{1}{2}A}$$

and also from the triangle CBE ,

$$\frac{BE}{BC} = \frac{b + c}{a} = \frac{\sin(C + \frac{1}{2}A)}{\sin \frac{1}{2}A}.$$

Equating these values of $(b + c)/a$ there results $\sin(B + \frac{1}{2}A) = \sin(C + \frac{1}{2}A)$. Therefore, $B = C$, and in like manner it may be shown that $A = B$. Hence the three angles of a triangle are equal to each other. Where lies the fallacy?

92

The computations of trigonometry are generally made by the help of logarithmic tables, but it is not wise that beginners should use logarithms at the start. Natural functions to three places are all that are needed in the blackboard work of a school and with such tables the angles of a triangle can be computed to the nearest quarter-degree from given sides. For example, when the two sides a and b , and their included angle C are given, the best formulas to use for finding the angles B and A are

$$\cot B = \frac{a/b}{\sin C} - \cot C, \quad \cot A = \frac{b/a}{\sin C} - \cot C.$$

For example, let $a = 225$ feet, $b = 350$ feet, $C = 48^\circ 15'$. Here $a/b = 0.643$, $b/a = 1.556$, $\sin C = 0.746$, $\cot C = 0.893$; then $\cot B = -0.031$, $\cot A = 1.193$, whence $B = 91^\circ 45'$ and $A = 40^\circ 0'$. This is certainly close enough for blackboard work.

93

There are interesting relations between the sides of a triangle when the angles are related in value. Thus when angle A is double the angle B we have $a^2 = b^2 + bc$. When angle A is three times the angle B , then we have $bc^2 = (a + b)(a - b)^2$.

94

In the American Mathematical Monthly for March, 1914, Artemas Martin finds many triangles with integral sides,

one angle being 60° . Thus the sides 8, 3, 7 give a triangle in which the angle opposite 7 is 60° ; so also the sides 8, 5, 7 give such a triangle. Only two other triangles can here be mentioned: 15, 8, 13, and 15, 7, 13.

95

ANALYTIC TRIGONOMETRY

The deductions of the formulas expressing relations between the functions of angles is often called analytic trigonometry. For instance $(2 \cos \theta)^5 - 5 (2 \cos \theta)^3 + 5 (2 \cos \theta) = 2 \cos 5 \theta$ is a formula for the quintisection of an angle. Also $(2 \cos \theta)^3 - 3 (2 \cos \theta) = 2 \cos 3 \theta$ is the formula for the trisection of an angle. When 5θ or 3θ is given $\cos \theta$ can be algebraically expressed, but such expressions cannot be handled numerically since they contain the square root of a negative quantity. Among the many interesting formulas of higher trigonometry the following deserve special consideration:

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} +, \text{etc.} \quad (1)$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} +, \text{etc.} \quad (2)$$

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} +, \text{etc.}$$

In these formulas the values of θ are to be taken in radians; thus, for an angle of 30° the value of θ is $1/6 \pi$, and for an angle of θ° the value of θ is $\pi\theta^\circ/180$. In the last formula e denotes the number $2.71828 \dots$ or the base of the Napierian system of logarithms. The symbol (!) denotes the product of the natural numbers; thus: $4! = 1 \times 2 \times 3 \times 4 = 24$.

Let i denote the imaginary $\sqrt{-1}$, and in the last formula change θ to $i\theta$; then it becomes

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \text{etc.}$$

This may be written in the form

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

or

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (3)$$

Similarly, replacing θ by $-i\theta$ there is found

$$e^{-i\theta} = \cos \theta - i \sin \theta. \quad (4)$$

Adding these two equations and also subtracting the second from the first, there results

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \quad i \sin \theta = \frac{1}{2} (e^{i\theta} - e^{-i\theta}), \quad (5)$$

which are remarkable expressions for the sine and cosine in terms of the imaginary $\sqrt{-1}$. These wonderful formulas are due to the great mathematician Euler. What do these formulas mean?

96

COMPLEX QUANTITIES

Equation (3) forms a basis for the extensive branches of vector analysis and quaternions, for $\cos \theta + i \sin \theta$ is what is called a complex number, the general expression for which is $a + ib$. To define and understand $a + ib$ it is first necessary to understand i . By the rules of simple algebra it is found that:

$$i^2 = (\sqrt{-1})^2 = -1, \quad i^3 = (\sqrt{-1})^3 = -\sqrt{-1} = -i,$$

$$i^4 = (\sqrt{-1})^4 = +1, \quad i^5 = (\sqrt{-1})^5 = +\sqrt{-1} = +i.$$

Now the following graphic representation agrees with these

results. In Fig. 33 let a line be drawn from O toward the right to represent $+1$ and one of equal length be drawn to the left to represent -1 . Also let a line be drawn upward from O to represent $+i$ and one downward to represent $-i$. Now let multiplication by i indicate turning a line of unit length through 90 degrees about the axis O . Then $+1 \times i = +i$, or the line $+1$ has been turned into the position shown by $+i$ in the figure. Also $+i \times i = -1$ or the line $+i$ has been turned to the position -1 ; also $-1 \times i = -i$, and $-i \times i = +1$. Thus with this graphic representation we see at once that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$ and $i^5 = +i$.



Fig. 33

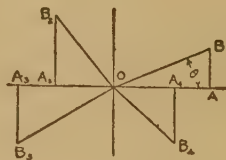


Fig. 34

Now to explain $a + ib$, let OA in Fig. 34 be laid off to the right to represent $+a$, and then at A lay off AB at right angles to OA to represent $+ib$. Then $OA + AB$ represents $a + ib$, or in other words $a + ib$ locates the point B . But the shortest way to go from O to B is by the hypotenuse OB , or by the vector addition $OA + AB = OB$. Here OB may be called a vector and be indicated in general by R , so that $R = a + ib$ is the vector which locates a point B , this point being located either by going directly to it by the shortest distance R , or by stepping off a units from O toward the right and then b units upward. When a is negative and b positive a point B_2 in the second quadrant

is located; when both a and b are negative a point B_3 in the third quadrant is located; when a is positive and b negative a point B_4 in the fourth quadrant is located.

The complex quantity $a + ib$ is frequently more conveniently expressed by

$$R = r (\cos \theta + i \sin \theta),$$

in which r is the length of the vector R , while $r \cos \theta$ and $r \sin \theta$ represent a and b . If this R is squared it becomes

$$R^2 = r^2 (\cos 2\theta + i \sin 2\theta),$$

and if it be raised to the n th power it becomes

$$R^n = r^n (\cos n\theta + i \sin n\theta),$$

which is known as the theorem of De Moivre. When $r = 1$ all the vectors are of unit length, and from the above formula (3) it is seen that

$$R = e^{i\theta} \quad \text{or} \quad e^{i\theta} = \cos \theta + i \sin \theta.$$

Hence $e^{i\theta}$ may be regarded as any radius in a circle of radius unit, this radius making an angle θ with the positive axis OA . The line OB in Fig. 35 represents such a vector: if this be squared OB revolves to the left through another angle θ and takes the position OB_1 .

From the last equation several remarkable algebraic expressions may be derived:

For $\theta = \frac{1}{2} \pi$,	$e^{\frac{1}{2}i\pi} = +i.$
For $\theta = \pi$,	$e^{i\pi} = -1.$
For $\theta = \frac{3}{2} \pi$,	$e^{\frac{3}{2}i\pi} = -i.$
For $\theta = 2\pi$,	$e^{2i\pi} = +1.$



Fig. 35

These are wonderful expressions, for e is the number 2.71828; algebraically or numerically they seem incomprehensible, but by the above graphic method of representation they are clearly understood.

SPHERICAL TRIGONOMETRY

The Greek astronomers developed and used spherical trigonometry long before plane trigonometry was known. These scientists were led to study the spherical triangle because it was necessary in the solution of problems involving the altitudes, azimuths, and hour angles of the stars and planets. The first tables were those of the chords of the angles. Later the Hindoos introduced the sine instead of the chord. Then the Arabs further developed the theory of both plane and spherical triangles.

The sum of the angles of a spherical triangle is always greater than 180 degrees, and the excess over 180 degrees depends on the area of the triangle. A rough rule for the spherical triangles measured in geodetic surveys is that there is one second of spherical excess for each 76 square miles of area. The same rule applies to spherical polygons. Thus, a triangle or polygon of the size of the state of Connecticut has a spherical excess of about 64 seconds. A trirectangular triangle which covers one-eighth of the surface of the earth has a spherical excess of 90 degrees, for the sum of its angles is 270 degrees.

When two plane triangles are equal one can always be made to coincide with the other, either by motion along the plane or by turning it over on one of the sides as an axis. But there can be two spherical triangles which have their sides and angles equal each to each, and yet it is impossible by any kind of motions to bring them into coincidence. They must, however, be called equal, since every part of one is equal to a corresponding part in the other and their areas are the same. Here is a case where equal things cannot coincide or be imagined to coincide.

98

HYPERBOLIC TRIGONOMETRY

Since 1875 there has been developed the interesting subject called Hyperbolic Trigonometry. This has nothing whatever to do with triangles, but it is intimately connected with a rectangular hyperbola. In the circle of Fig. 36 let P be any point on the circle and from it let the perpendicular PS be dropped upon the radius OA ; also at A let the perpendicular AT be drawn until it meets the radius OP produced. Let AOP be the angle θ , then if the radius OA is unity, OS is $\cos \theta$, SP is $\sin \theta$, AT is $\tan \theta$, and OT is $\sec \theta$.

In the right-hand diagram of Fig. 36 let OA be the semi-major axis of an equilateral hyperbola, and P any point on

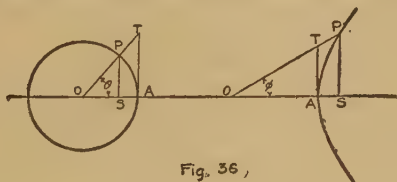


Fig. 36,

the curve. From P drop PS upon OA produced, and from A draw AT perpendicular to AO until it meets OP . Then if AO is unity, OS is called the hyperbolic cosine, SP the hyperbolic sine, AT the hyperbolic tangent, and OT the hyperbolic secant. Let ϕ be double the area of the hyperbolic sector OAP , then

$OS = \cosh \phi$, $SP = \sinh \phi$, $AT = \tanh \phi$, $OT = \operatorname{sech} \phi$, where $\cosh$ means hyperbolic cosine, $\tanh$ means hyperbolic tangent, and so on. In academic slang these are often pronounced cosh, shin, than, shec.

In the circle $\overline{OP}^2$ equals $\overline{OS}^2 + \overline{SP}^2$, or $\cos^2 \theta + \sin^2 \theta = 1$. In the equilateral hyperbola $\overline{OP}^2$ equals $\overline{OS}^2$ minus $\overline{SP}^2$ or

$\cosh^2 \phi - \sinh^2 \phi = 1$. The letter θ denotes an angle or a multiple or submultiple of π . The letter ϕ denotes a number which may have any value.

The formulas (1) to (5) in No. 95 are true whether θ be real or imaginary. Changing θ to $i\theta$ in formula (5) and then replacing $\cos i\theta$ by $\cosh \theta$ and $-i \sin i\theta$ by $\sinh \theta$, they become

$$\cosh \theta = \frac{1}{2} (e^{\theta} + e^{-\theta}), \quad \sinh \theta = \frac{1}{2} (e^{\theta} - e^{-\theta}), \quad (6)$$

which are the values of the hyperbolic cosine and sine in exponential form. Here as always, the letter e denotes the number 2.71728, the base of the hyperbolic system of logarithms. Let these expressions be squared and the second subtracted from the first, then $\cosh^2 \theta - \sinh^2 \theta = 1$, which agrees with the equation established in the preceding paragraph, θ being here the double of a hyperbolic sector.

The computation of the cosine and sine of an angle cannot be made by the formulas (5) since there is no way of obtaining the imaginary power $e^{i\theta}$. But the computation of the hyperbolic cosine and sine is easily made from (6); for instance let $\theta = 2$, then $e^2 = 7.389057$ and $e^{-2} = 0.135335$, hence $\cosh 2 = 3.626861$.

As the quantity θ varies from 0 to ∞ , $\cosh \theta$ varies from 1 to ∞ and $\sinh \theta$ from 0 to ∞ , while $\tanh \theta$ varies from 0 to 1. There is no periodic repetition of the real values of the hyperbolic functions as is the case with the circular functions, but yet they have imaginary periods.

Enough has now been stated to give the young reader a glimpse of the fundamental ideas at the foundation of hyperbolic functions. But, he may ask, of what use or importance are they? The reply to this is, that such functions constantly turn up in the solution of practical prob-

lems. For example, take the catenary, which is the curve assumed by a cord or cable suspended from two points and hanging freely under its own weight. Let y be the ordinate or height of any point of the curve above a certain horizontal plane, x the distance of the point to the right or left of the lowest point of the curve and c a certain constant, then the equation of the curve is

$$y = \frac{1}{2} c (e^{x/c} + e^{-x/c}) = c \cosh x/c.$$

This equation in terms of e was deduced a hundred years or more before hyperbolic cosines were ever thought of, but the second form in terms of $\cosh$ has many practical advantages over it.

Hyperbolic functions also turn up in the theory of arches, in the formula for a beam subject to both flexure and tension, in the construction of charts to represent large portions of the earth's surface, and especially in the electrical discussion of alternating currents. The length and areas of many curves which were formerly stated in terms of Napierian logarithms are now more conveniently expressed by hyperbolic functions.

CHAPTER V

ANALYTIC GEOMETRY

99



DESCARTES, a French philosopher, who lived in the first half of the seventeenth century, devised the method of coördinates by which curves can be graphically represented and their properties be studied through their equations. In this method, as applied to plane curves, two lines called axes are drawn at

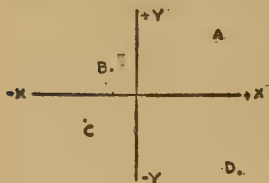


Fig. 37

right angles. Their intersection O is called the origin of coördinates. Values of x are laid off parallel to the X -axis, and values of y parallel to the Y -axis. Positive values of x are laid off to the right of the Y -axis and negative ones to its left. Positive values

of y are laid off upward from the X -axis and negative ones downward. Thus if a point has the coördinates $x = 3$, $y = 2$, it is located at A ; if it has the coördinates $x = -1$, $y = 1$, it is located at B ; if it has $x = -2$, $y = -1$, it is at C ; and if it has $x = 4$, $y = -3$, it is at D .

100

The equation of a curve is an equation which gives the relation between the coördinates of any point on the curve. Thus $x/4 + y/3 = 1$ gives the relation between x and y for

every point; if $y = 3$ then $x = 0$, if $y = 6$ then $x = -4$, if $y = -3$ then $x = 8$. Plotting these three points by laying off their coördinates from the X - and Y -axes, it is seen that they are on one straight line. When $y = 0$ the line crosses the X -axis at $x = 4$; when $x = 0$ the line crosses the Y -axis at $y = 3$; and its inclination to the X -axis is the slope of 3 to 4. Thus any equation of the first degree between two variables is the equation of a straight line. This line is of infinite length for no matter how great y is taken the corresponding value of x can be found from the equation.

The equation of a circle is $x^2 + y^2 = r^2$ and that of an equilateral hyperbola is $x^2 - y^2 = r^2$. Equations of many other curves are known to the student who reads these pages. By discussion of these curves we learn their shape, we draw tangents to them at given points, we find where two curves intersect, and later by the help of the calculus we can find their lengths and also determine the areas included between them and the axes.

101

Coördinates are used also for the graphic representation of statistics and natural phenomena. For example, let the following be the means of the mean monthly temperatures at a certain place for a series of years:

Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
20°	28°	28°	34°	47°	60°	78°	77°	66°	47°	38°	25°

If the reader will place twelve points at equal distances apart on the X -axis, and then lay off upward the temperatures given, 20 for Jan. at the first point, 28 for Feb. at the second point, and so on, and then connect these points with a curve, he will have a graphic representation of the mean

monthly temperatures throughout the year at the given place. This curve resembles the curve of sines, that is the curve whose equation is $y = \sin x$. In plotting the points through which the curve is to be drawn, it is best to use paper that is ruled into squares.

A point to be plotted is sometimes indicated by the notation $(5, 3)$ where the first number is the coördinate x and the second the coördinate y . Let the reader plot the following ten points, numbering them 1, 2, 3, etc., and then join the points by a curve in the order of the numbers: $(1.2, 2.0)$, $(2.0, 1.1)$, $(3.0, 1.0)$, $(4.0, 1.7)$, $(4.1, 2.6)$, $(3.0, 3.1)$, $(2.0, 3.5)$, $(2.1, 4.6)$, $(3.0, 5.0)$, $(3.8, 4.6)$.

102

Let it be required to find the intersection of the circle $x^2 + y^2 = 16$ with the straight line $3x + 5y = 15$. By roughly plotting the circle and the line there will be found the two points $(3.9, 0.6)$ and $(-1.4, 3.8)$. But closer values can be found by finding the values of x and y by combining the two equations; this gives 3.95 and -1.31 for the two values of x , and 0.63 and 3.79 for the corresponding values of y .

Now let it be required to find the intersection of the circle $x^2 + y^2 = 16$ with the straight line $x + y = 8$. Plotting the circle and the line it is seen that they do not intersect. But combining the two equations there are found for x the two values $4 + \sqrt{-8}$ and $4 - \sqrt{-8}$, and for y the corresponding values $4 - \sqrt{-8}$ and $4 + \sqrt{-8}$. These imaginary values show, of course, that the straight line does not intersect the circle. But have they no other meaning? Yes, they have a definite geometric meaning which will be explained lat

Let it be required to plot the curve whose equation is

$$\pm x = \frac{1}{2}y - a\sqrt{1 - \left(\frac{y-a}{a}\right)^{80}}.$$

Assume values of y and find the corresponding values of x ; thus, when y is negative then x is imaginary, when $y = 0$ then x is 0, when y is very small then $x = \pm a$, when $y = \frac{1}{2}a$ then $x = \pm \frac{3}{4}a$, when $y = a$ then $x = \pm \frac{1}{2}a$, when $y = 3/2a$ then $x = \pm \frac{1}{4}a$, as y approaches $2a$ then x approaches 0, when $y = 2a$ then $x = \pm a$, when y is greater than $2a$ then x is imaginary. Accordingly, Fig. 38 represents the real curve. The key-note to the formation of this equation lies in the quantity $(y-a)/a$ which is raised to the 80th or any other large even power. When this quantity is numerically less than unity its 80th power is very small; for example, let $y = a/10$, then the fraction is -0.9 and its 80th power is 0.00021 . This subtracted from 1 gives 0.99979 the square root of which is almost 1. Thus on any practicable scale of plotting, Fig. 38 is a proper representation of the equation.



Fig. 38

103

TRANSCENDENTAL CURVES

Curves involving circular functions are often called transcendental. Like the sine curve, $y = \sin x$, they have a period 2π and hence repeat themselves in both directions to infinity. For example, the curve whose equation is $\sin^2 y = \sin x \sin \frac{1}{4}x$ consists of the series of points, ovals, and lemniscates shown in Fig. 39. Here both x and y are taken in radians. When $x = \pi$ or 2π then $\sin^2 y = 0$,

$\sin y = 0$, or $y = 0, \pi, 2\pi$, etc., and this gives the vertical column of points. When $y = \pi$ or 2π then either $\sin x = 0$ or $\sin \frac{1}{4}x = 0$, whence $x = \frac{1}{2}\pi, \frac{3}{2}\pi$, etc., or $2\pi, 6\pi$, etc. When $\sin x \sin \frac{1}{4}x$ is negative then y is imaginary. Thus with much labor the curves in Fig. 39 are constructed.

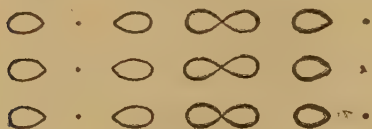


Fig. 39

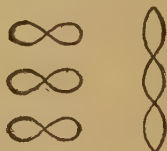


Fig. 40

When the equation is $\sin^2 y = \sin x \sin \frac{1}{5}x$ some parts of the diagram undergo great changes and Fig. 40 results. Many beautiful diagrams of such curves, constructed by Newton and Phillips, may be seen in the Transactions of the Connecticut Academy for 1875.

The curve $\tan^2 x + \tan^2 y = 100$ gives a series of squares spaced like the plan of a rectangular city.

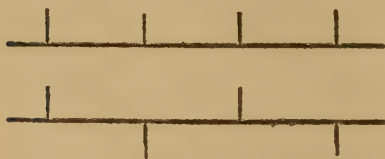


Fig. 41

are taken in degrees, each side of a square is $168^\circ 36'$ and the width of the streets between the squares is $11^\circ 24'$.

The equation $y = \sin^{80} x$ gives a diagram like the upper one in Fig. 41 while the equation $y = \sin^{81} x$ gives the lower one. Both of these extend right and left to infinity.

The equation $\sin^2 \sqrt{x^2 + y^2} = 0$ is satisfied only when $\sqrt{x^2 + y^2} = 0^\circ, 180^\circ, 360^\circ, 540^\circ$, etc. When $\sqrt{x^2 + y^2} = 180^\circ$ then $x = 180^\circ$ and $y = 144^\circ$ or $x = 144^\circ$ and $y = 108^\circ$.

When $\sqrt{x^2 + y^2} = 360^\circ$ then $x = 216^\circ$ and $y = 288^\circ$ or $x = 288^\circ$ and $y = 216^\circ$. Hence the equation represents a series of points radiating from the origin, each point enjoying the property that its x , y , and r form the sides of a right-angled triangle which are in the ratio 3 : 4 : 5, or in the ratio 4 : 3 : 5.

The equation $y^2 = \sin x$ represents a series of circles in a horizontal row.

The equation $y^2 = x^2$ represents two intersecting straight lines which are inclined at angles of 45° to the rectangular axes. But the equation $y^2 = x^2 \sin x$ represents a series of ovals included between such lines as the envelope. Also the equation $y^2 = ax$ represents the common parabola, but the equation $y^2 = ax \sin x$ represents a series of ovals included within the parabola as an envelope.

104

POLAR COÖRDINATES

When a point is located by its distance from an origin O and the angle θ which that distance, or radius vector, makes with the horizontal X -axis, the case is one of polar coördinates. Any equation in rectangular coördinates may be transformed into one in polar coördinates by replacing x by $R \cos \theta$ and y by $R \sin \theta$, where R is the radius vector. Thus the parabola $y^2 = ax$ is expressed in polar coördinates by $R = a/\tan \theta \sin \theta$.

105

The equation $R = 10 + 10 \cos \theta$ gives an interesting curve called the cardioid. When $\theta = 0^\circ$ then $R = 20$, when $\theta = 60^\circ$ then $R = 15$, when $\theta = 90^\circ$ then $R = 10$, when $\theta = 120^\circ$ then $R = 5$, when $\theta = 180^\circ$ then $R = 0$.

This gives the upper half of the curve and the lower half is similar to it. This curve is entirely appropriate for a young man to send to a young woman in the month of February as a mathematical valentine, provided she understands polar coördinates.

The equation $R = 10 + 20 \cos \theta$ is called the trisectrix, because, after the curve is drawn it may be used for the trisection of an angle. For explanation see the *Encyclopedia Britannica*.

The equation $R = 5 \pm \sqrt{\sin 20^\circ \theta}$ represents twenty small circles arranged around a central point. When $\theta = 0^\circ$ then $R = 5$, when $\theta = 4.5^\circ$ then $R = 5.7$ or 4.3 , when $\theta = 9^\circ$ then $R = 5$, when θ is between 9° and 18° then R is imaginary; when $\theta = 18^\circ$ then $R = 5$ and another small circle begins.

The equation $R = 10 \pm \sin \theta$ represents another curious curve, the discussion of which will be left to the ingenuity of the reader.

106

REMARKS

Paradoxes and fallacies decrease as we ascend the mathematical ladder. Under Arithmetic we had a dozen or two, under Algebra nearly as many, under Geometry a few, under Trigonometry only one. In Analytic Geometry H. E. Licks tried hard to find a fallacy, but without success.

The curves formed by the intersection of a plane with a right cone are called conic sections. If twenty persons, not mathematicians, are asked what curve is formed when the plane is parallel to the base of the cone, they will reply that it is a circle. If they are asked what is the curve when the plane is slightly inclined to the base, they will reply that

it is an oval with the larger end nearest the base; this answer is also generally given by those who have studied Analytic Geometry and have rarely thought of the subject since their college days. Yet this curve is an ellipse which is no blunter at the lower end than at the upper, and Analytic Geometry furnishes a rigorous proof of it. The curve is always a true ellipse until the plane has the same inclination as the side of the cone, then it is a parabola. For greater inclinations the curve is a hyperbola. Hence, the circle, ellipse, parabola, and hyperbola are called the conic sections.

Analytic Geometry does not furnish an array of interesting facts. To draw a tangent to a curve at a given point is not an interesting exercise, nor is the determination of the angle included between two lines. The important things in Analytic Geometry seem to be, (1) the plotting of curves from given coördinates, (2) the plotting of curves from their equations, and (3) the construction of an equation to represent a series of plotted points. None of these receive proper attention in the text-books; indeed, the last is rarely or never alluded to.

Formal rules for solving certain problems are liked by teachers, but in reality they introduce a repression of reasoning which is harmful to the student. For example, in the American Mathematical Monthly for April, 1914, there is given the following as the "usual rule" for drawing a line through a given point parallel to a given line. "Step 1: Change the constant term in the given equation to k . Step 2: Substitute the coördinates of the given point in the equation of step 1 and solve for k . Step 3: Substitute this value of k in the equation of step 1, and the result is the desired equation." Now what real knowledge of Analytic Geometry has the student gained by this procedure? He

has gained discipline in obeying orders but no development in his reasoning powers has resulted. If this is the way the subject is taught, it is indeed a case of the blind leading the blind.

On each blackboard of a school where this subject is taught there should be in one corner a permanent ruling of about a hundred squares each one inch in size. When the problem is given to draw a line through a point parallel to a given line, the student should put rectangular axes on this network of squares, then plot the given point, next draw the given line by finding the points where it intersects the axes, then draw freehand a parallel line through the given point. He will then see that the two lines have the same slope, and it should be left to his own ingenuity to find an equation to represent the second line. Two exercises like this do more to develop the reasoning powers of the student than a hundred exercises solved by formal rules with "steps."

107

GRAPHIC SOLUTION OF EQUATIONS

Let it be required to solve the cubic equation $x^3 + ax + b = 0$. This equation can be supposed to result from the elimination of y from the two simultaneous equations $y = x^3$ and $y = -ax - b$. The curve whose equation is $y = x^3$ can be plotted on squared paper by the help of a table of cubes, then if the straight line whose equation is $y = -ax - b$ is drawn on this plot the abscissas of its points of intersection with the curve are the roots of the equation $x^3 + ax + b = 0$. When the line intersects the curve in only one point, the equation has one real root and two imaginary roots; when it intersects the curve in three points there are three real roots. Thus let $x^3 - 20x +$

$30 = 0$ be the given equation. In Fig. 42 the full line represents the equation $y = x^3$ and the broken line the equation $y = 20x - 30$, the horizontal scale being twenty times the vertical scale. Here there are three real roots $x_1 = +1.8$, $x_2 = +3.3$, $x_3 = -5.1$.

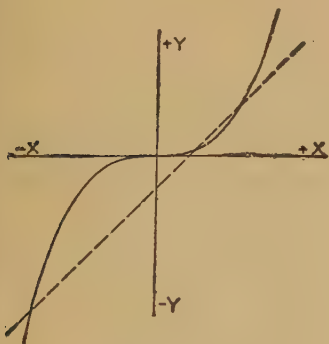


Fig. 42

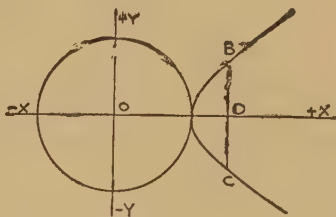


Fig. 43

The cubic equation $y^3 + Ay^2 + Bx + C = 0$ can be reduced to the form $x^3 + x^2 + bx + c$ by putting $y = A^{\frac{1}{3}}x$ and then dividing by A . Here the curve whose equation is $y = x^3 + x^2$ can be plotted by the help of tables of cubes and squares, then the points where the straight line $y = -ax - b$ intersects the curve give the roots of the cubic equation.

Let it be required to find the points where the straight line $x = 5$ intersects the circle $x^2 + y^2 = 16$. The algebraic solution gives $y = \pm 3\sqrt{-1}$ and these values may be represented graphically in the following manner. Let the hyperbola $x^2 - y^2 = 16$ be drawn, with its vertex at A , in a plane at right angles to that of the axes XX and YY . Let $OD = x = 5$, then DB and DC are each equal to 3 with

respect to the hyperbola but each is equal to $3\sqrt{-1}$ with respect to the circle. Many interesting illustrations of this method of representing the imaginary and complex roots of equations may be seen in *Graphic Algebra* by Phillips and Beebe.

CHAPTER VI

CALCULUS

108

 SIR ISAAC NEWTON'S method of obtaining the differentials of quantities is given in Section II, Book II, Lemma II of his great Principia. It is remarkable that in this discussion there is no reference to infinitesimal quantities, or to the vanishing of their powers and products, or to the doctrine of limits. The discussion is clear and simple, and well adapted to the comprehension of a beginner in calculus. Hence space may well be taken here to explain his method. The reader may, perhaps, most conveniently refer to the American edition of Newton's Principia, or Principles of Natural Philosophy, published at New York in 1848, where the discussion begins on page 262.

Newton supposes the sides A and B of a rectangle to vary from smaller to larger values. When the sides had the values $A - \frac{1}{2}a$ and $B - \frac{1}{2}b$, its area was $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. When the sides are increased to $A + \frac{1}{2}a$ and $B + \frac{1}{2}b$, the area becomes $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From this latter rectangle subtract the former and there remains $aB + bA$. "Therefore," says Newton, "with the increments a and b of the sides, the increment $aB + bA$ of the rectangle is generated." In the modern notation x and y are taken as the variable sides of the rectangle and dx and dy as their increments or differentials, then in the above dis-

cussion replacing A by x and B by y , also a by dx and b by dy , there is found $x \cdot dy + y \cdot dx$ as the differential or increment of the area xy . Thus $dxy = x \cdot dy + y \cdot dx$ is deduced without considering dx and dy to be infinitely small. In the following derivation of differentials modern notation will be used instead of that employed by Newton.

Numerous algebraic functions can be differentiated by use of the formula $dxy = x \cdot dy + y \cdot dx$. To find the differential of x^2 make $y = x$ and $dy = dx$; then $dx^2 = 2 x dx$. To find the differential of x^3 , make $y = x^2$ and $dy = 2 x dx$ in the formula; then $dx^3 = 3 x^2 dx$. To find the differential of x^4 , make $y = x^3$ and $dy = 3 x^2 dx$ in the formula; then $dx^4 = 4 x^3 dx$. From these, by induction, it is concluded that $dx^n = nx^{n-1} dx$, and, as in the case of the binomial formula, we infer correctly that this applies whether n be a whole number or a fraction, whether it be positive or negative. Thus $dx^{-1} = -x^{-2} dx$, and $dx^{n/m} = (n/m)x^{n/m-1} dx$.

Also in the formula $dxy = x \cdot dy + y \cdot dx$, the letters x and y may indicate any variable quantities. Thus to find the differential of the fraction u/v let $x = 1/v$, $dx = -dv/v^2$, $y = u$, $dy = du$ in the formula, then $du/v = (1/v) du - (u/v^2) dv$. This may be written in the form $(v du - u dv)/v^2$, that is, the differential of a fraction equals its denominator into the differential of its numerator minus its numerator into the differential of its denominator, divided by the square of the denominator.

MAXIMA AND MINIMA

109

The differential calculus enables easy the solution of many problems involving maxima and minima. For example, a tin cylindrical box of diameter a and height h is to

be made to contain Q cubic inches of material. If the thickness of the tin is t what must be the ratio of the height to diameter in order that the least amount of tin may be used? Here the quantity of tin is $(\pi ah + \frac{1}{2} \pi a^2) t$, this including the cover of the box; also $Q = \frac{1}{4} \pi a^2 h$. Taking the value of h from the second equation and substituting it in the first gives $(Q/a + \frac{1}{2} \pi a^2) t$ as the expression for the quantity to be made a minimum. Placing the derivative of this equal to zero and solving, there results $a^3 = 4Q/\pi$ or $Q = \frac{1}{4} \pi a^3$ and equating this to the above value of Q , there is found $h = a$. Hence, for minimum material the height of the box must be equal to its diameter.

110

As a second example, let it be required to find the length of the longest straight stick AB which can be put up a circular shaft in the ceiling of a room, the height of the room being h and the diameter of the shaft a . Here it is convenient to let θ be the angle which the stick makes with the floor; then $AB = h/\sin \theta + a/\cos \theta$. Placing the derivative equal to zero, there results $\tan \theta = (h/a)^{\frac{1}{3}}$. Then expressing $\sin \theta$ and $\cos \theta$ in terms of $(h/a)^{\frac{1}{3}}$ there is found for the length of the stick $AB = (h^{\frac{2}{3}} + a^{\frac{2}{3}})^{\frac{3}{2}}$. This is a simple way to solve a problem which has proved a stumbling block to many.

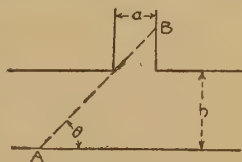


Fig. 44

111

Hundreds of problems similar to the above may be found in books and mathematical journals, hence H. E. Licks gives one not found in books, namely, to determine the path

of a ray of light from a source S to the eye at E , when a transparent glass plate is interposed between them. Let

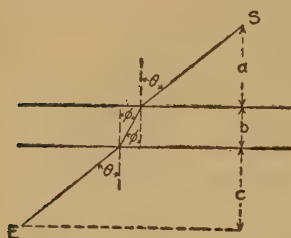


Fig. 45

Fig. 45 show the path by the heavy broken line, the light moving in straight lines both within and without the plate, as is known by experiment. Let a , b , c , d be the distances between S and E measured normal and parallel to the plate. Let θ be the angle which the ray makes

with the normal to the plate before it enters and after it leaves, and ϕ the angle which it makes with that normal within the plate. Let v_1 be the velocity of the light without the plate and v_2 the velocity within it. Then the time required to travel from S to E is

$$t = a \sec \theta / v_1 + b \sec \phi / v_2 + c \sec \theta / v_1.$$

Also the quantities are connected by the geometric relation $a \tan \theta + b \tan \phi + c \tan \theta = d$. Now the path must be such as to make the time t a minimum. Hence, if N is a constant to be determined, the quantity

$t = (a + c) \sec \theta / v_1 + b \sec \phi / v_2 + N [(a + c) \tan \theta + b \tan \phi - d]$ is to be made a minimum. Differentiating there is found

$$\frac{dt}{d\theta} = (a + c) \sec \theta \tan \theta / v_1 + N (a + c) / \cos^2 \theta = 0,$$

$$\frac{dt}{d\phi} = b \sec \phi \tan \phi / v_2 + N b / \cos^2 \phi = 0.$$

From the first of these $N + \sin \theta / v_1 = 0$ and from the second $N + \sin \phi / v_2 = 0$, whence by elimination of N , there is found $\sin \theta / \sin \phi = v_1 / v_2$. Hence the ratio of the sines of the angles made by the ray with the normal of the plate is

equal to the ratio of the velocities of light without and within the plate. Thus the path is completely determined in terms of the velocities v_1 and v .

The ratio of $\sin \theta / \sin \phi$ is, in optics, called the index of refraction and its values have been accurately determined by measurements for different materials. Thus when light passes from air into water this index is 1.33, that is, the velocity of light in water is about three-fourths of its velocity in air.

112

THE CELL OF THE HONEY BEE

The cell made by the bee in which to store honey is shown in Fig. 46. The end $ABDE$ is the top of the cell which is closed with a plane cap after the cell is filled with honey. The cross-section of the cell is a regular hexagon formed with thin sides of wax. The bottom of the cell $abdefg$ is terminated by three equal planes which meet at the apex c and which are rhomboidal in shape so as to form a depressed cup, for the points a, f, d are further away from the top of the cell than are the points b, e, g , and the apex c is still further away. The angles of these rhomboids at b, e, g are equal to the angles at c . If a cross-section of the cell be taken anywhere on its length, there results a hexagon each of whose interior angles is 120° , but the six angles in the bottom of the cell at b, e, g, c are only about 110° owing to the inclination of the three planes.

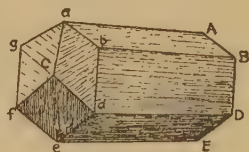


Fig. 46

It is evident that there is a certain inclination of these planes which will give less material for the cell than if the lower end were made plane like the upper end. To deter-

mine this inclination is a problem in maxima and minima which has received much attention because the conclusion

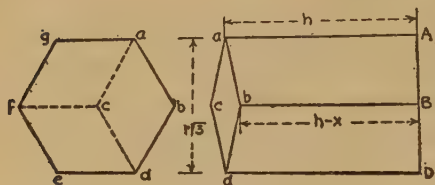


Fig. 47

deduced agrees closely with the actual construction of the cell. Fig. 47 gives end and side views of the cell. Let h be the mean length of the cell, and $h - x$ the length of

the side Bb . Regarding $abdefg$ as the cross-section let each of its sides be called r , then bc is also r ; but by virtue of the inclination of the rhombus $abdc$ the distance bc becomes increased as shown in Fig. 48. Let this increased distance be called t . The

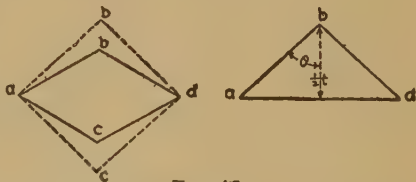


Fig. 48

plane $abdc$ in Fig. 47 then has an inclination such that the distance t is $\sqrt{r^2 + 4x^2}$.

Now considering the amount of wax in the cell to be proportional to the sum of the areas of its sides and bottom, the expression for the total area in terms of x is made a minimum. The area of the two sides shown in the side view is $r(2h - x)$, the area of the inclined rhombus $abdc$ is $\frac{1}{2}t \times r\sqrt{3}$ or $\frac{1}{2}r\sqrt{3}\sqrt{r^2 + 4x^2}$, and hence the total area A to be made a minimum is

$$A = 3 \left[r(2h - x) + \frac{1}{2}r\sqrt{3}\sqrt{r^2 + 4x^2} \right].$$

Differentiating this value of A with respect to x , equating the derivative to zero, and solving, gives $x = r\sqrt{1/8}$ for the

value of x which renders the quantity A a minimum. For this value of x the area A becomes $A_1 = 3r(2h + \frac{1}{2}r\sqrt{2})$ which is proportional to the amount of wax in the sides and bottom of one cell.

If the cell had a plane bottom at right angles to the sides, the area of the sides and bottom is found by making $x = 0$ in the above expression for A , whence $A_0 = 3r(2h + \frac{1}{2}r\sqrt{3})$.

The ratio of A_0 to A_1 now is

$$s = \frac{2h + \frac{1}{2}r\sqrt{3}}{2h + \frac{1}{2}r\sqrt{2}} = \frac{4h/r + \sqrt{3}}{4h/r + \sqrt{2}}$$

and the following are values of this ratio for various values of h/r :

For $h/r = 0$,	$s = 1.225$.
For $h/r = 1$,	$s = 1.072$.
For $h/r = 2$,	$s = 1.034$.
For $h/r = 4$,	$s = 1.018$.
For $h/r = 6$,	$s = 1.013$.

It hence appears that the saving in wax of the actual cell over a cell with plane bottom is 7.2 per cent when $h = r$, 3.4 per cent when $h = 2r$, and 1.8 per cent when $h = 4r$. The height of the cell is usually between $h = 2r$ and $h = 5r$, so that the saving in wax is on the average about 2 per cent.

Early writers on this problem paid great attention to the angles abd and acd of the rhombus in Fig. 47. The tangent of the angle θ (Fig. 48) when x has the value $r\sqrt{1/8}$ which renders A a minimum, is $\frac{1}{2}r\sqrt{3}/\frac{1}{2}t$, and since $t = r\sqrt{3/2}$ this tangent is $\sqrt{2}$. Then by the help of a logarithmic table it is easy to find θ , and its double $109^\circ 28' 16''$ is the angle abd in the inclined rhombus, and this is also the value of each of the angles at the apex c of the pyramidal cup.

Statements were made that this angle had been measured and found to be $109^{\circ} 28'$, from which it was concluded that the cell of the bee agreed most closely with that which theory demanded for the minimum quantity of wax. However, evidence regarding these measurements is wanting, and indeed it would be a very difficult matter to measure this angle to such a degree of exactness.

This problem first received discussion in the eighteenth century, and writers on it generally extolled the wonderful instinct of the bee in adopting a form of cell which led to economy in wax. The production of wax is an exhausting

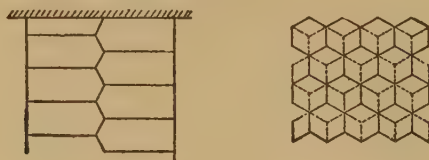


Fig. 49

operation for the bee, and moreover sixteen pounds of honey are needed to produce one pound of wax. Economy is hence promoted by any method which will limit the production of wax to the least possible amount. According to most writers the bee has solved this problem in a most ingenious mathematical manner, and its instinct should be regarded as one of the most remarkable in nature.

In order to judge how far these high encomiums are justified, it is necessary to examine the construction of the honeycomb. An inspection of one shows that it is formed by two tiers of horizontal cells with their bases resting on a vertical midrib in which the pyramidal cups are formed. In Fig. 49 the heavy lines of the right-hand diagram give a front view of the cells on one side of the midrib, and the broken lines

show the cells on the other side. These two tiers of cells alternate in a curious manner, the bottom of one cell abutting against the bottoms of three cells of the other tier. The cells themselves are either horizontal or inclined very slightly upward, and the left-hand diagram in Fig. 49 shows a vertical section before they are filled with honey. Both tiers of the comb are supported by the central midrib which is attached to the ceiling of the hive. In building the cells the bees begin at the top and work downward, the base of each cell being of course built before the cell itself.

The examination of such a honeycomb will also show that the midrib forming the bases of the cells is thicker than the walls of the cell itself, this probably being so because it is required to carry all the weight of the cells and honey. In fact it has been stated that the midrib is thicker near its top than lower down. The observations of the writer lead to the rough conclusion that the midrib is $1\frac{1}{2}$ or 2 times thicker than the walls of the cells. This being the case, the above theory falls to the ground as fallacious.

Let n be the ratio of the thickness of the midrib to that of the walls of the cell. Then the above expression for the area A becomes

$$A = 3r \left(2h - x + \frac{1}{2}n \sqrt{3r^2 + 12x^2} \right).$$

The value of x which renders this a minimum is now found to be $x = \frac{1}{2}r/\sqrt{3n^2 - 1}$. From this the following values are found for the angle abd of the inclined rhombus and for the angles at the apex c :

For $n = 1$,	$abd = 109^\circ 28' 16''$.
For $n = 1\frac{1}{2}$,	$abd = 116^\circ 40' 0''$.
For $n = 2$,	$abd = 117^\circ 59' 10''$.
For $n = 4$,	$abd = 119^\circ 38' 58''$.

Here the last value is given in order to show that the bottom of the cell becomes practically flat when the midrib is four times as thick as the walls of the cell. For a perfectly flat bottom this angle is of course exactly 120° .

While this variation in thickness of the midrib appears to take the problem outside of the domain of pure mathematics, yet such is not really the case. Exact observations of the way in which the bees build the comb are needed, as also measurements of the bases of the cells, and perhaps these may be made in the laboratories of natural history. At present the writer offers the following as conjectures: (1) that the cell of the bee is built according to the rules deduced above for minimum material when the midrib is equal in thickness to the walls of the cell, (2) that this shape of the cell is not due to an instinct for securing the minimum quantity of wax, but is entirely due to a method of construction which arises from a necessity that the bees in adjoining cells should crowd together as closely as possible.

The first conjecture can only be established by measurements made on the same midrib at both upper and lower parts of the comb, and on different midribs in different kinds of cells. If the angles of the inclined faces of the apex c of the pyramidal cup can be measured, the writer predicts

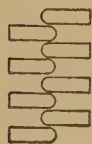


Fig. 50

that these will approximate to the value $109^\circ 28' 16''$.

Concerning the second conjecture it should be noted that the midrib is built by bees which face each other in the work as shown in the left-hand diagram of Fig. 50. In

order that the midrib between the two tiers of cells may be properly compacted it is necessary that the heads of the

bees in one tier should alternate with the heads of those in the other tier. In the right-hand diagram the full-line circles show the heads of the bees in one tier and the broken-line circles the heads of those in the other tier. Here it is seen that each bee occupies a triangular position between three other bees, and with this arrangement it is indispensably necessary that the bottom of each cell should be a pyramidal cup having three sides.

In the theoretic cell deduced at the beginning of this article the inclination of each of the three planes of the bottom of the cell to a cross-section is such that the tangent of the angle is $2x/r$ or $\sqrt{\frac{1}{2}}$. This corresponds to $35^{\circ} 45' 52''$. The first conjecture of the author demands that this inclination should always be $35^{\circ} 45' 52''$ whatever be the thickness of the midrib. Further investigation of these three planes will show that the diedral angle between any two is exactly 120° , and this in the conjecture of the writer is always closely the case.

Let the reader take four spheres of equal size, lay three of them on a table so that each is tangent to the other two, and then put the fourth sphere upon these three. If these points be located and three tangent planes be drawn, it is a simple matter of computation to find that each plane makes an angle of $35^{\circ} 45' 52''$ with the horizontal, and that the diedral angle included between any two of the planes is 120° . Thus a series of alternating spheres gives the same planes as are found in the cells of the bee; hence one cause of the inclination of the latter is undoubtedly the alternating heads of the bees in forming the midrib shown in Fig. 50.

The reason why the cells are hexagonal has often been discussed. All writers are in agreement that this is due to

the circumstance that each cell is surrounded by six others, and that if any other form than the hexagonal were adopted vacant spaces would be left, which could not be filled with honey. Moreover it is thought by the writer that any cell wall must be built by bees working upon both sides of it. Now in the hexagonal cell the diedral angle between any two adjacent side walls is 120° . At the base of this cell, as we have seen, the diedral angle between any two of the planes forming the pyramidal cup is 120° ; also each of these planes in intersecting a side of the hexagonal cell makes with it an angle of 120° . Hence in the bee cell every diedral angle is 120° . The angles at the top of the cell, where the cap is put on, are not here included; but as long as the bee is in the cell she has only to deal with diedral angles of 120° .

The conclusion of this discussion is that the cells of the bee are not built from any instinct for reducing the production of wax to a minimum, but rather from the necessity that their heads must alternate in forming the midrib in order to properly compact it. This necessity results in planes inclined to each other at angles of 120° . Perhaps it may be said that the bee has an instinct to build planes inclined at this angle, but more properly, it seems to the writer, it may be said that the work of the bees is more easily done in this way than in any other. Economy in labor rather than in material appears to lie at the foundation of the symmetric form of the cell of the industrious honey bee.

An interesting critical article by Glaisher will be found in the London Philosophical Magazine for August, 1873, where the history of this famous problem is set forth in full detail. At that date the belief appears to be undoubted that the form of the cell is due to an instinct of the bees for saving as

much wax as possible, and this is referred to as one of the most remarkable instances of instinct in nature. Since the discussion here given indicates otherwise, further investigations are in order to fully solve the problem, and these are only possible after many observations and measurements have been made in entomological laboratories.

113

INTEGRAL CALCULUS

Differentiation is a definite process and any given function of a single variable can be differentiated. But there is no way to integrate except from a knowledge of what has been done in differentiation. In this respect the two branches of calculus are analogous to involution and evolution in arithmetic; any given number may be raised to a stated power, but when the power is given there is no way to find the root except by guess work and trial. There are about twenty-five fundamental integrals which are known to be correct because the differentiation of them furnishes the given expression with which we start. All the rest of integral calculus consists in reducing the quantity to be integrated to one of the fundamental forms.

For instance, $\int x^{n-1} dx$ equals x^n/n because the differential of the latter is $x^{n-1} dx$ and for no other reason. Similarly $\int \sin x \cdot dx$ equals $-\cos x$ because the differential of $\cos x$ is $-\sin x \cdot dx$. In all cases the correctness of an integral is to be determined by differentiating it.

114

To the above statement that any function can be differentiated there seems one exception. Weirstrass has devised

a certain series, expressed in symbolic form, for which a derivative cannot be obtained, because in any interval, no matter how small, there are an infinite number of bends of the curve, so that at any given point it is not possible to draw a tangent to the curve. This expression, however, is little more than a curiosity to a beginner.

115

When $y \cdot dx$ is required, y being expressed in positive integral powers of x , then the integral can be directly found from the formula

$$\int y \cdot dx = yx - D_1 \frac{x^2}{2!} + D_2 \frac{x^3}{3!} - D_3 \frac{x^4}{4!} + D_4 \frac{x^5}{5!} - \text{etc.},$$

in which D_1, D_2, D_3 , are the first, second, and third derivatives of y with respect to x . For example, let $y = ax^2 + x^3$, then $D_1 = 2ax + 3x^2$, $D_2 = 2a + 6x$, $D_3 = 6$, $D_4 = 0$. Then, substituting in the formula, there is found

$$\int (ax^2 + x^3) dx = 1/3 ax^3 + 1/4 x^4.$$

Unfortunately this formula does not seem to apply to other functions which have no D equal to 0 but in each given case we are forced to consult a catalog of integrals, or to reduce the given function to one whose integral is known.

116

John Phoenix, the first real humorist of America, was a graduate of West Point and hence well versed in mathematics. In his essay called "Report of a Scientific Lecture," he alludes to the importance of adding a constant to the result of an integration. He says:

By a beautiful application of the differential theory the singular fact is demonstrated, that all integrals assume the forms of the atoms of which they

are composed, with, however, in every case the important addition of *a constant*, which like the tail of a tadpole, may be dropped on certain occasions when it becomes troublesome. Hence, it will evidently follow that space is round, though, viewing it from various positions, the presence of the cumbersome addendum may slightly modify the definiteness of the rotundity. To ascertain and fix the conditions under which, in the definite consideration of the indefinite immensity, the infinitesimal incertitudes, which, homogeneously aggregated, compose the idea of space, admit of the computable retention of this constant, would form a beautiful and healthy recreation for the inquiring mind; but, pertaining more properly to the metaphysician than to the ethical student, it cannot enter into the present discussion.

117

LENGTHS OF CURVES

The lengths of nearly all curves are expressed in terms of circular, hyperbolic, or logarithmic functions. Thus, the length of an arc of a circle is always in terms of π , and the length of an arc of a parabola is in terms of a hyperbolic logarithm. The story is told that a German professor, lecturing to his class two hundred years ago, said that the length of no curve could be algebraically expressed, and that the next day one of the students brought to him the derivation of the length of an arc of the semi-cubical parabola in algebraic terms. This curve has the equation $n^{\frac{1}{2}}y = x^{\frac{3}{2}}$. The derivative dy/dx is $\frac{3}{2} n^{-\frac{1}{2}} x^{\frac{1}{2}}$ and the length of an arc between the limits $x = 0$ and $x = a$ is:

$$\int_0^a dx \sqrt{\frac{9x}{4n} + 1} = \frac{8}{27} n \left[\left(\frac{9x}{4n} + 1 \right)^{\frac{3}{2}} \right]_0^a = \frac{8}{27} n \left[\left(\frac{9a}{4n} + 1 \right)^{\frac{3}{2}} - 1 \right].$$

For example, let the equation of the curve be $y = x^{\frac{3}{2}}$, then $n = 16$, and the length of the arc between the limits of $x = 0$ and $x = a = 4$ is $122/27 = 4.5189$. Whether the story is true or not, the length of this curve can certainly be algebraically expressed and be computed by simple arithmetic.

118

Another curve whose length is expressed by simple algebra is the cycloid. This curve is generated by a point on the circumference of a wheel which rolls along the straight line DE . Thus the point A in Fig. 51 reaches the

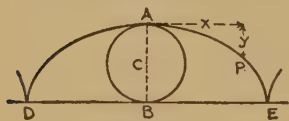


Fig. 51

horizontal line at E when the circle has made half a revolution and in its progress the semi-cycloid APE is described. Let a be the radius CA of the generating circle, and P be any point on the cycloid

whose coördinates are x and y , the latter being measured downward. Then the length of the curve AP is $\sqrt{8ay}$ and the length of AE is $8a$, expressions of the greatest simplicity. The area between the cycloid DAE and the straight line DE is three times the area of the generating circle or $3\pi a^2$. The cycloid has also interesting properties which will be mentioned later under Mechanics.

119

The lengths of some curves of pursuit are also algebraically expressible. The simplest case (Fig. 52) is where the hare starts at O and runs with uniform speed v on the axis OY while the dog starts at a point A on the X -axis and runs always directly toward the hare with the speed V . When the dog is at P the hare is at Q and the tangent to the curve of pursuit is PQ . Let a be the distance between the initial positions O and A , and let n be the ratio of the speeds

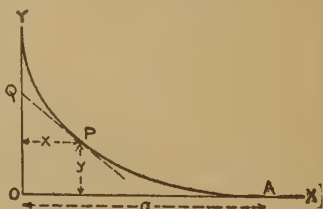


Fig. 52

initial positions O and A , and let n be the ratio of the speeds

v/V . Then the equation of the curve of pursuit, when n is not equal to unity, is

$$y = \frac{a^n x^{1-n}}{2(n-1)} + \frac{x^{1+n}}{2a^n(n+1)} + \frac{an}{1-n^2}.$$

For example, let the dog run twice as fast as the hare, or $n = \frac{1}{2}$, then the equation of the curve is

$$y = \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} - a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{2}{3}a.$$

The length of an element of the curve being $dx \sqrt{1+p^2}$, where $p = dy/dx$, the length of the curve from A to P is found to be algebraically expressed, thus:

$$\int_x^a \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \frac{4}{3}a - a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}}.$$

When $x = 0$ then $y = 2/3 a$, and the length of the curve is $4/3 a$. Here the dog has run double the distance that the hare has run, and it catches the hare at the point $x = 0$, $y = 2/3 a$.

When n is equal to or greater than unity, the dog can never catch the hare. When n is less than unity the dog will catch the hare. The student in calculus may find it profitable to solve the following problem: Let the dog run 10 feet per second and the hare 8 feet per second, and let $a = 720$ feet; prove that the dog will catch the hare in 6 minutes and 40 seconds from the instant when the hare starts at O and the dog starts at A .

CHAPTER VII

ASTRONOMY AND THE CALENDAR

120

STRONOMY is probably the most ancient of the physical sciences, the first facts being observed by shepherds who watched their flocks at night. The historian Josephus, in his *Antiquities of the Jews*, begins with the creation of the world and follows closely the biblical narrative. Speaking of Phaleg, fourth in descent from Noah and of his son Tera, who was the father of Abraham, he says: "God afforded them a longer life on account of their virtue and the good use they made of it in astronomical and geometrical discoveries." Speaking of the sojourn of Abraham among the Egyptians, he says, "He communicated to them arithmetic and delivered to them the science of astronomy; . . . they were unacquainted with those parts of learning, for that science came from the Chaldeans into Egypt and from thence to the Greeks."

121

The order of the twelve constellations of the zodiac may be remembered by the following ancient lines:

The Ram, the Bull, the Heavenly Twins,
Next the Crab, the Lion shines,
The Virgin, and the Scales,
The Scorpion, Archer, and the Goat,
The man who holds the watering Pot,
And Fish with glittering tails.

In memorizing this it is well to note, that the word shines should rhyme with Twins, and Pot with Goat.

The order here is from west toward east; when the Ram is setting in the west the Scales are rising in the east, when the Scales are setting in the west the Fish are rising in the east. This is a rough statement only, for at certain seasons of the year less than one-half of these constellations are above the horizon, while at other seasons more than one-half of them are visible at one time. Unfortunately the artist who put several of the constellations of the zodiac on the ceiling of the grand concourse in the Grand Central Station in New York, reversed this order, for there we see Aquarius in the east while the Crab is in the west. The copy from which he worked evidently had been incorrectly made; perhaps he took it from a celestial globe and then turned it around so as to interchange east and west.

122

The greatest of all optical instruments was the reflecting telescope of William Herschel, which was finished in 1789. The tube was forty feet in length, five feet in diameter, and weighed 60,000 pounds. With this telescope, magnifying 6450 times, he discovered two new moons circling around the planet Saturn, and recorded hundreds of new double stars and nebulæ. His sister, Caroline Herschel, was his constant companion in all his astronomical labors.

William Herschel died in 1822. In 1839 his celebrated son, John Herschel, took down the great telescope, which had then become a victim to the ravages of time and could no longer be used. The long tube was carefully laid upon three stone pillars where it could be preserved as a relic of the past. In the Christmas holidays of that year, John

Herschel, his wife, and their six children held a family feast in the great tube, and there they sang a song written by him in honor of the occasion:

In the old telescope's tube we sit,
And the shades of the past around us flit,
His requiem we sing with shout and din
As the old year goes out and the new year comes in.
Merrily, Merrily, let us all sing,
And make the old telescope rattle and ring.

Full fifty years did he laugh at the storm,
And the blast could not shake his majestic form.
Now prone he lies where he once stood high
And searched the heavens with his broad bright eye.
Merrily, Merrily, etc.

Here watched our father the wintry night
And his gaze was fed by pre-adamite light;
His labors were lighted by sisterly love,
And united they strained their vision above.
Merrily, Merrily, etc.

123

Galileo was the first man who looked at the heavenly bodies through a telescope. It was in 1610 that he saw four satellites moving around the planet Jupiter, and this demolished the theory that the earth was the center around which the planets revolved. These four satellites of Jupiter were the only ones known until 1892, but since then four smaller ones have been discovered. The earth has one moon, Mars has two, Jupiter has eight, and Saturn has nine or ten. The inner moon of Mars is near the planet and has such a high velocity that it rises in the west and sets in the east, while both new and full moon can be observed in a single night. All other known satellites, like our own moon, rise in the east and set in the west.

124

BOTANY AND ASTRONOMY

If we examine the leafy stem of a plant we shall find the leaves upon it arranged in a symmetrical order and in a way uniform for each species. If a line be drawn around the stem from the base of one leaf stalk to that of the next, and so on, this line will wind around the stem as it rises, and on any particular plant there will be the same number of leaves for each turn around the stem. In the basswood, the Indian corn, and all the grasses, we have the two-ranked arrangement; the second leaf starting on exactly the opposite side of the stem from the first, the third opposite the second and hence directly over the first, so that all the leaves are in two vertical ranks, one on one side of the stem and one on the other. Next is the three-ranked arrangement such as is seen in sedges; here the second leaf is one-third of the way around the stem, the third one two-thirds, and the fourth one directly over the first. Then in the apple, cherry, and most of our common shrubs, the leaves are arranged in five vertical ranks, and the spiral winds twice around the stem before it reaches a leaf directly over the first one; here the distance between any two ranks is two-fifths of the circumference of the stem. Then in the common plantain there are eight ranks, and three turns around the stem, so that the distance between any two ranks is three-eighths of the circumference.

Now if we express these arrangements by figures, we have the fractions $1/2$, $1/3$, $2/5$, $3/8$, in which the denominator expresses the number of ranks and the numerator the number of turns of the spiral line around the stem before it reaches a leaf directly above the one from which it started.

Thus $1/2$ stands for the two-ranked arrangement where there are two turns. But we notice that the numerator of any fraction is equal to the sum of the numerators in the two preceding fractions, and that the same is true for the denominators. Then the next fraction after $3/8$ will be found by adding 2 and 3 for its numerator and 5 and 8 for its denominator, which gives $5/13$. Thus we have the following series, $1/2$, $1/3$, $2/5$, $3/8$, $5/13$, $8/21$, $13/34$, etc., and just such arrangements of leaves are found, and no others. The fraction $5/13$ gives the law for the common house leek, the others are found in the pine family and in many small plants.

The furthestmost planet from the sun is Neptune, then follow Uranus, Saturn, Jupiter, the Asteroids, and Mars, then the Earth, Venus, and Mercury. Neptune makes its revolution around the sun in about 60,000 days, Uranus in 30,000 days or $1/2$ the time of Neptune; in like manner Saturn's period is nearly $1/3$ of that of Uranus, Jupiter's period is $2/5$ that of Saturn, and so on until we come to the earth, following closely the same series as given above for the leaves on a stem. Thus the mathematical expression of the arrangement of the leaves of plants is approximately the same as that of the periods of the exterior planets. These arrangements of leaves ensure to plants a better distribution of the light and heat of the sun; the periods of the planets render them stable under the laws of gravitation. Perhaps the botanist, had he known that these figures apply both to leaves and planets, might have foretold the discovery of the asteroids or announced the existence of Neptune.

125

THE MOON HOAX

In 1833 Sir John Herschel sailed from England for the Cape of Good Hope, carrying a large telescope with which to view the southern stars. This was before the times of steamboats and telegraphs so that more than two years passed away before any definite account of the discoveries of Sir John reached England or America. In 1835 the *New York Sun* published a series of articles, entitled "Great Astronomical Discoveries made by Sir John Herschel at the Cape of Good Hope." In the first article was given a circumstantial and highly plausible account as to how this early and exclusive information had been obtained by the paper. Then comes an interesting account of the inception and construction of the great telescope which he carried to the Cape. The idea of great magnifying power originated, it was said, in a conversation with Sir David Brewster regarding optics. "The conversation became directed to that all-invincible enemy, the paucity of light in powerful magnifiers. After a few minutes silent thought Sir John diffidently inquired whether it would not be possible to effect a transfusion of artificial light through the focal object of vision. Sir David, somewhat startled at the originality of the idea, paused awhile, and then hesitatingly referred to the refrangibility of rays and the angle of incidence. Sir John continued, 'why cannot the illuminated microscope, say the hydro-oxygen, be applied to render distinct, and if necessary even to magnify the focal object?' Sir David sprang from his chair in an ecstasy of conviction, and leaping halfway to the ceiling exclaimed, 'Thou art the man!'"

The interest of the reader being thus aroused by this

imaginary scientific conversation, the article goes on to describe the great telescope which was shipped to the Cape and there drawn by two relief teams of oxen to the place where it was erected. This place was "a perfect paradise in rich and magnificent mountain scenery, sheltered from all winds and where the constellations shone with astonishing brilliancy." Here Sir John observed stars and nebulae, but above all he paid particular attention to the Moon. The magnifying power of his telescope was 42,000 times, so that objects on the Moon could be seen as if only six miles away and an object only 18 inches in diameter could be plainly recognized. Hence he clearly saw on the moon "basaltic rock, forests, and water, beaches of brilliant white sand girt with castellated marble rocks." He beheld herds of brown quadrupeds of the bison kind, each animal having a hairy veil over its eyes, and he conjectured "that this was a providential contrivance to protect the eyes from the great extremes of light and darkness to which all beings on the moon are periodically subjected." He also saw a species of beaver which was acquainted with the use of fire as was evident from the smoke that occasionally rose from their habitations.

Finally, of course, his search was rewarded by the sight of human beings with wings and who walked erect and dignified when they alighted on the plain. "They appeared in our eyes scarcely less lovely than the representations of angels by our more imaginative schools of painters; their works of art were numerous and displayed a proficiency of skill quite incredible to all except actual observers."

This hoax was immediately swallowed by the general public and caused much discussion. The *Sun* issued in pamphlet form an edition of 60,000 copies which were sold

in less than a month, and translations of it were made in Europe. In 1859 a second pamphlet edition was issued in New York with illustrations of the moon and with added notes.

The author of this most entertaining and successful hoax was Richard Adams Locke, then editor of the *Sun*. He was engaged in newspaper work for a large part of his life and died in 1871 at his home on Staten Island. An obituary notice describes him as "a warm-hearted man, well read, enthusiastic, and sometimes very eloquent on paper. His habits were rather convivial, but he was just and fearless, full of the best intentions, and overflowing with original inspirations."

126

The planet on which we live claims, of course, a large share of our attention. In 1878 Americus Symmes published his "Theory of Concentric Spheres," demonstrating that the earth is hollow, habitable within, and widely open about the poles. It contains arguments drawn from the statements of explorers in the polar regions, from the dip and variation of the magnetic needle, from the migrations of fish, from the spots on the sun and from the rings of Saturn. According to this remarkable theory, there are two openings at the poles into the hollow earth, the diameter of the northern one being about 2000 miles while the southern one is somewhat larger; the planes of these openings are parallel to each other but they make an angle of 12 degrees with the equator. Capt. Symmes imagined that the crust of the earth is about a thousand miles in thickness but he wisely refrained from giving any account of what is found within the hollow sphere.

127

Perhaps the earliest mention of a sun dial is that found in II Kings, xx, 9-11:

9. And Isaiah said, This sign shalt thou have of the LORD, that the LORD will do the thing that he hath spoken; shall the shadow go forward ten degrees or back ten degrees?

10. And Hezekiah answered, It is a light thing for the shadow to go down ten degrees: nay, but let the shadow return backward ten degrees.

11. And Isaiah the prophet cried unto the LORD: and he brought the shadow ten degrees backward, by which it had gone down in the dial of Ahaz.

On a properly constructed sun dial, such as is described below, the shadow cannot go backward. But a dial having a vertical style or gnomon when tilted from the horizontal possesses the property that the shadow will travel backward for a short time near sunrise and sunset. A dial with a vertical gnomon is, however, quite useless in telling the time of day.

128

THE SUN DIAL

Four or five hundred years ago the only way to tell the hour of the day was by looking at a sun dial, for clocks and watches had not then come into use. Fig. 53 shows such a dial, which indicates 2 P. M. by the edge of the shadow cast upon the graduated surface by an inclined gnomon. Of course the sun dial is useless on a cloudy day, but when the sun does shine it gives apparent



Fig. 53

solar time with a probable error of about ten minutes, which is sufficiently close for the purposes of agriculture. A sun

dial is usually placed in a horizontal plane, but in olden times they were often put on the walls of churches and public buildings, and many such can be seen in Europe even at this day.

The board on which the lines of the sun dial are drawn may be of any shape, but in Fig. 54 it is indicated as rectangular. This board is to be placed horizontally with its central line *NS* coinciding with the meridian of the place and is usually observed from

its southern side. The shadow of the gnomon *AB* falls toward the western side in the morning and toward the eastern side in the afternoon. The lines which radiate from the center *A* being properly drawn the observer will see the

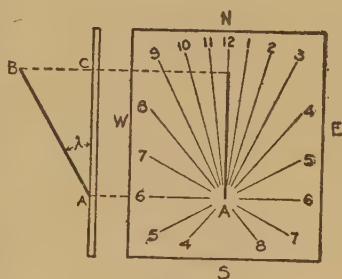


Fig. 54

shadow coinciding with the line 8 at eight o'clock in the morning, with the line 12 at noon, and with the line 3 at three o'clock in the afternoon. When the shadow is one-fourth of the distance from the line 3 to the line 4, the sun time is 3.15 P. M.

The gnomon *AB* must be inclined to the plane of the board at an angle equal to the latitude of the place. Sometimes this is a thin sheet of metal *ABC* fastened onto the board, the edge *AB* being the true gnomon; sometimes it is a small metal rod *AB*, the end *B* being supported by another rod *BC*. It is essential that the inclination of *AB* to a horizontal dial plate must be equal to the latitude of the place, or for a dial in any position *AB* must point to the celestial pole.

How to make a horizontal dial: On a smooth board draw the lines NS and EW . The northern end of the line NS is to be numbered 12 for twelve o'clock noon, and the ends EW are to be numbered 6 for 6 A. M. and 6 P. M. The latitude of the place may be taken from a good map with sufficient precision for the construction of a sun dial, or if great precision is required it may be found by an astronomical observation. This latitude λ is to be used for constructing the gnomon, and also for computing the angles which the radiating lines of the dial make with the central line NS .

To find the angle α which any radiating line makes with the central line AN , let n be the number of hours before or after noon when the shadow should fall on that line; then

$$\tan \alpha = \sin \lambda \tan n 15^\circ.$$

Accordingly, the values of $\tan \alpha$ are as follows:

For 1 and 11 o'clock, $n = 1$ and $\tan \alpha = 0.268 \sin \lambda$.

For 2 and 10 o'clock, $n = 2$ and $\tan \alpha = 0.577 \sin \lambda$.

For 3 and 9 o'clock, $n = 3$ and $\tan \alpha = 1.000 \sin \lambda$.

For 4 and 8 o'clock, $n = 4$ and $\tan \alpha = 1.732 \sin \lambda$.

For 5 and 7 o'clock, $n = 5$ and $\tan \alpha = 3.732 \sin \lambda$.

For 6 and 6 o'clock, $n = 6$ and $\tan \alpha = \infty$.

For 7 and 5 o'clock, $n = 7$ and $\tan \alpha = -3.732 \sin \lambda$.

Now the values of α will be different for different latitudes. The following table gives values of α for three latitudes which have been computed from the above formulas with the help of a trigonometric table.

	$\lambda = 30$	$\lambda = 40$	$\lambda = 50$
For 1 and 11 o'clock	$\alpha = 7^\circ 38'$	$9^\circ 46'$	$11^\circ 36'$
For 2 and 10 o'clock	$\alpha = 16 \quad 06$	$20 \quad 22$	$23 \quad 52$
For 3 and 9 o'clock	$\alpha = 26 \quad 33$	$32 \quad 44$	$37 \quad 27$
For 4 and 8 o'clock	$\alpha = 40 \quad 54$	$48 \quad 04$	$53 \quad 00$
For 5 and 7 o'clock	$\alpha = 61 \quad 49$	$67 \quad 23$	$70 \quad 43$
For 6 and 6 o'clock	$\alpha = 90 \quad 00$	$90 \quad 00$	$90 \quad 00$
For 7 and 5 o'clock	$\alpha = 118 \quad 11$	$112 \quad 37$	$109 \quad 17$

When the radiating lines have been drawn, the gnomon put in place, and the board neatly painted, the sun dial is ready for erection. The board must be placed duly level with its *NS* line coinciding with the true meridian, and then, when the sun shines, delighted spectators may compare apparent solar time with their watches and wonder at the scientific skill of the youth who constructed the sun dial.

The largest sun dial ever built is at the royal observatory in Jaipur, India; it was erected about 1750 by the Mahá-rája Siwái Jai Singh II. Its gnomon is about 175 feet long and this can be ascended by stairs. The shadow of the gnomon falls on a large stone quadrant of 50 feet radius along which it moves at the rate of $2\frac{1}{2}$ inches per minute. Jaipur is in latitude 27 degrees north.

129

Southworth, in his "Four Thousand Miles of African Travel" (New York, 1875) gives a novel method of determining the true meridian: "The Arab when he prays, kneels toward Mecca. It is said that even the youngest never fails to bend, almost accurately, in that direction. Thus, in the form of living flesh, we had the Arab, by whom to find the variation of the compass; and, with the corrected bearing, we could find, when the sun bore due south or otherwise, the true meridian, and consequently noon."

130

The civil day begins at sunset among the Mahomedans and at midnight in Christian countries and is divided into twenty-four hours. The sun dial has been used from a remote antiquity to indicate apparent solar time. Clocks with wheels were devised about 1250 but they did not come

into general use until after 1600. It was found that these clocks at some times of the year were slower and at other times faster than apparent solar time. An accurate clock or watch keeps mean solar time, this being the time which would be indicated on a sun dial if the sun were perfectly uniform in his apparent motion throughout the year. The difference between apparent and mean solar time is called the Equation of Time and its values are given in some almanacs under the headings "clock slow" or "clock fast." The following table shows such values to the nearest minute which are to be added to apparent time (or subtracted when marked $-$) in order to give mean or clock time.

1 Jan.	3 min.	1 May	-3 min.	1 Sept.	0 min.
10 Jan.	8 min.	10 May	-4 min.	10 Sept.	3 min.
20 Jan.	11 min.	20 May	-3 min.	20 Sept.	-6 min.
1 Feb.	14 min.	1 June	-2 min.	1 Oct.	-10 min.
10 Feb.	14 min.	10 June	-1 min.	10 Oct.	-13 min.
20 Feb.	14 min.	20 June	1 min.	20 Oct.	-15 min.
1 Mar.	12 min.	1 July	4 min.	1 Nov.	-16 min.
10 Mar.	11 min.	10 July	5 min.	10 Nov.	-16 min.
20 Mar.	8 min.	20 July	6 min.	20 Nov.	-14 min.
1 Apr.	4 min.	1 Aug.	6 min.	1 Dec.	-11 min.
10 Apr.	1 min.	10 Aug.	5 min.	10 Dec.	-7 min.
20 Apr.	-1 min.	20 Aug.	3 min.	20 Dec.	-2 min.

This table will be useful when one compares his watch with a sun dial. As all the affairs of life are now regulated by clock time, it also explains why the time of sunset appears to rapidly become earlier in October and to rapidly become later in January.

The apparent and mean solar time above described is different for places having different longitudes, and in general may be designated as local time. In recent years, owing to the requirements of railroad operation, most

clocks and watches keep standard time or the local time on a certain meridian. In the United States there are four standard meridians, those of longitude 75° , 90° , 105° , and 120° west of Greenwich. Eastern standard time is mean solar time of the 75° meridian, central standard time is mean solar time of the 90° meridian, mountain standard time is mean solar time of the 105° meridian, and Pacific standard time is mean solar time of the 120° meridian. In going from one of these meridians to the next one, our watch must be set one hour backward or forward according as we go west or east.

When a watch keeping standard time is read at a place which is one degree of longitude west of the standard meridian it is four minutes faster than mean local time of that place; when the place is two degrees to the westward the watch is eight minutes faster, for three degrees westward twelve minutes faster and so on. When read at places to the eastward it is four minutes slower for each degree of longitude. Hence, this must be taken into account also when comparing a watch with a sun dial.

131

DAYS, MONTHS, AND YEARS

Julius Cæsar, with the help of the astronomer, Siosene-ges, introduced the method of reckoning known as the Julian calendar. The year being 365.2422 solar days, he took 365 such days for a common year and 366 days for a leap year, so that the average length of a year was 365.25 days. This Julian calendar is still in use in Russia and Greece, but it was supplanted in most of Europe in 1582 by the Gregorian calendar. In the Julian calendar all years divisible by 4 were leap years; in the Gregorian calendar years

divisible by 4 are leap years unless they are divisible by 100 and not by 400. Thus, in the Gregorian calendar the years 1600 and 2000 are leap years, but the years 1700, 1800, 1900 are common years. In 1582 the Julian calendar was ten days slower than the Gregorian, after 1700 it became eleven days slower, and since 1900 it has been thirteen days slower. Hence, Jan. 1, 1917 of the Gregorian calendar corresponds to Dec. 19, 1916 of the Julian.

1752

September hath XIX Days this Year.

First Quarter, the 15th day at 2 afternoon.
Full Moon, the 23rd day at 1 afternoon.
Last Quarter, the 30th day at 2 afternoon.

M D	W D	Saints' Days Terms, &c.	Moon South	Moon Sets	Full Sea at Lond.	Aspects and Weather
1	f	Day br. 3.35	3 A 27	8 A 29	5 A 1	Π 2 ♀
2	g	London burn.	4 26	9 11	5 38	Lofty winds

According to an act of Parliament passed in the 24th year of his Majesty's reign and in the year of our Lord 1751, the Old Style ceases here and the New takes its place; and consequently the next Day, which in the old account would have been the 3d is now to be called the 14th; so that all the intermediate nominal days from the 2d to the 14th are omitted or rather annihilated this Year; and the Month contains no more than 19 days, as the title at the head expresses.

14	e	Clock slo. 5 m.	5 15	9 47	6 27	HOLY ROOD D. and hasty showers
15	f	Day 12 h. 30 m.	6 3	10 31	7 18	
16	g		6 57	11 23	8 16	
17	A	15 S. AFT. TRIN.	7 37	12 19	9 7	More warm and dry weather
18	b		8 26	Morn.	10 22	
19	c	Nat. V. Mary	9 12	1 22	11 21	
20	d	EMBER WEEK	9 59	2 24	Morn.	♂ ♀ ♂ Π 2 ♀ ♂ ⊙ ♂ ♂ ⊙ ♀
21	e	St. MATTHEW	10 43	3 37	0 17	
22	f	Burchan	11 28	D rise	1 6	
23	g	EQUAL D. & N.	Morn.	6 A 13	1 52	Rain or hail now abouts * 2 ♀
24	A	16 S. AFT. TRIN.	0 16	6 37	2 39	
25	b		1 5	7 39	3 14	
26	c	Day 11 h. 52 m.	1 57	8 39	3 48	
27	d	EMBER WEEK	2 56	8 18	4 23	
28	e	Lambert bp.	3 47	9 3	5 6	
29	f	St. MICHAEL	4 44	9 59	5 55	
30	g		5 43	11 2	6 58	

In Great Britain and its colonies the change of the Julian to the Gregorian calendar was not made until 1752. In September of that year eleven days were omitted from the almanacs. The above is a copy of the calendar for September, 1752, taken from the Almanac of Richard Saunders, Gent., published in London. All English and American almanacs gave similar statements for that month. The Ladies' Diary or Woman's Almanac indulged in poetry, appropriate to the occasion:

The third of September the fourteenth is nam'd,
For which British annals will ever be fam'd.
For by Wisdom and Art to the House made appear
The Sun was reduc'd to attend on the Year;
His Julian vagaries long time has he known,
But has now got a new bridal Year of his own

132

In both Julian and Gregorian calendars the months are those established by Julius Cæsar, namely:

Thirty days hath September,
April, June, and November,
All the rest have thirty-one,
Save in February which, in fine,
In common years, hath twenty-eight,
And in leap years twenty-nine.

The time when the year began has been different in different countries. In Cæsar's reign it appears that March was the first month; thus September was the seventh and December the tenth, as the names imply. The early English almanacs, however, begin the year with January as at present, but the legal year of the British government began on March 25, although March was called the first

month. In legal and church records prior to 1752, it is common to find dates like Feb. 20, 1695, or Feb. 20, 1695/6, these being intended for the historical or almanac year 1696.

133

A very convenient rule for determining the day of the week corresponding to the day of the month in any year was given by Prof. Comstock in *Science*, Nov. 18, 1898. Let Y be any year of the Gregorian calendar and D the day of the year. Divide $Y - 1$ by 4, by 100, and by 400, neglecting the remainder in each case. Then find S from

$$S = Y + D + \frac{Y - 1}{4} - \frac{Y - 1}{100} + \frac{Y - 1}{400}$$

and divide S by 7; the remainder gives the day of the week, 0 indicating Saturday, 1 Sunday, 2 Monday, and so on. For example, take July 4, 1916; here $Y = 1916$, $D = 186$, $(Y - 1)/4 = 478$, $(Y - 1)/100 = 19$, $(Y - 1)/400 = 4$. Then $S = 2565$, and this divided by 7 gives a remainder 3; hence, July 4, 1916 comes on Tuesday. The reason for this rule is clear, if it be remembered that all years exactly divisible by 4 are leap years except when they are even century years, as 1800, 1900, 2000, etc., when they must be divisible by 400; thus the subtractive term $(Y - 1)/100$ prevents the addition of an extra day during such years as 1800, 1900, and 2100, while it also makes only one extra day to be added during the year 2000. Of all the rules for finding the day of the week from a given day of the month and year, this is by far the simplest.

For the Julian calendar the following rule may be used to find the day of the week corresponding to a given date.

Let Y be the year and D the day of the year. Neglecting the remainder in the fractional term, compute S from

$$S = Y + D + \frac{Y - 1}{4} - 2$$

and divide S by 7; then the remainder gives the day of the week, 0 indicating Saturday, and so on. For example, Columbus discovered America on Oct. 12, 1492; here $Y = 1492$, $D = 286$, $(Y - 1)/4 = 372$, then $S = 2148$ which divided by 7 gives 6 for a remainder; hence America was discovered on a Friday. Again George Washington was born on Feb. 11, 1732; here $Y = 1732$, $D = 42$, $(Y - 1)/4 = 432$, and then $S = 2204$ which divided by 7 gives 6 for a remainder; hence, Washington was born on a Friday.

The common opinion that Washington was born on Feb. 22 is erroneous. This originated in the idea of irresponsible persons that Gregorian time ought to be extended backward into Julian time. This reprehensible idea is founded on no sound principle, and in celebrating the birthday of Washington on Feb. 22, we all commit grievous error.

134

J. W. Nystrom of Philadelphia devised about fifty years ago the "tonal system" of numeration in which 16 is the base instead of 10 as in the decimal system. The numerals 1, 2, 3, 4, etc., were called An, De, Ti, Go, etc., and new characters were devised for 10, 11, 12, 13, 14, 15. This system embraced also a new division of the year into 16 months, these having the names Anuary, Debrian, Timander, Gostus, Suvenary, Bylian, Ratamber, Mesidius, Nic-torary, Kolumbian, Husander, Victorious, Lamboary, Folian, Fylander, Tonborious, the first two letters of each

month being the names of the sixteen numerals. Nystrom certainly did his work well.

135

Josh Billings in his almanac said that the name February was derived from a Chinese word which meant kondem cold. Josh was right regarding the temperature. At the head of the calendar for July he gives this verse:

Young man, let hornets be
And don't go nigh the pizen snake too much,
For in the month of July
They a'in't healthy to the touch.

For another month he gives this excellent advice:

He who by farming would get rich,
Must plow, and hoe, and dig, and sich,
Work hard all day, and sleep hard all nite,
Save every cent and not get tite.

For the first of April he has the following:

April Phool was born this day,
A simpleton, but clever,
And though 3000 years of age,
He's just as big a phool as ever.

136

Comets in ancient times brought great mental distress upon people, for they were supposed to presage war, famine, or pestilence. Even to astronomers the phenomena of the tail being repelled by the sun backward from the nucleus of the comet has been a great mystery, for it seemed to contradict the law of universal attraction. Now, however, we understand that the small particles of the tail are driven away from the head by the pressure exerted by the light of the sun, so that the mystery appears to have been solved.

Yet even at this day the appearance of a comet incites a feeling of awe, and the words of the poet Holmes arise in the memory:

The Comet! He is on his way,
And singing as he flies;
The whizzing planets shrink before
The spectre of the skies.
Ah! well may regal orbs burn blue,
And satellites turn pale,
Ten million cubic miles of head,
Ten billion leagues of tail!

CHAPTER VIII

MECHANICS AND PHYSICS

137

IT IS a misfortune that physicists and engineers teach to students two different systems of units. A boy comes to a technical school, understanding perfectly, from his experience, what is meant by force and what is meant by a force of ten pounds or ten kilograms. The teacher of physics tells him that forces must be measured in poundals or dynes, notwithstanding that no apparatus for measuring forces in such units has ever been made or used. The result is great mental confusion to the boy, from which he does not recover until he joins the class in engineering where he finds that forces are measured in those units to which he had always been accustomed before he entered upon the instruction of the physicist. All this might be avoided if mechanics were omitted entirely from courses in physics. Surely the subjects of heat, light, sound, and electricity furnish a sufficient field for the physicist, without encroaching on the topic of mechanics, which properly belongs to the engineer.

138

The unfortunate equation $F = mf$ comes early in a course in mechanics as taught by a physicist. Here the mass m is measured in units of a standard lump of metal furnished by

the government; acting for one second on this lump is a force F , which produces the velocity f at the end of that second. More generally f is called the acceleration, or change in velocity in one second, and its unit is one unit of length per second. Let L represent length in general and T time, while M represents mass, then we have $F = ML/T^2$, or force dimensionally equals mass multiplied by length divided by the square of time. The student tries hard to comprehend this, but finds it impossible, for he *knows* that force is not ML/T^2 and he *knows* that there is no way to measure a force except by the number of units of force which it contains.

The truth of the matter is that the equation $F = mf$ is not true. Experiments and experience teach that mf is proportional to cF where c is a constant, not that mf equals F . When there are two different forces F and G which act at different times on the same body they produce accelerations f and g . Experience and experiments show that these forces are proportional to the accelerations which they produce, whence

$$F/G = f/g.$$

This is a fundamental equation which is entirely correct. If the teacher starts with this, his students will have no confusion of mind.

139

Into an apple cut two holes inclined like ab and cb in Fig. 55. Into each hole put a small quill so that when the string AC is inserted the friction may be small.

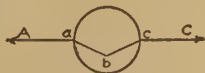


Fig. 55

Then pull horizontally upon the string by its ends A and C . As the pull increases, the apple will

be seen to rise vertically by the upward pressure of the string at b ; as the pull slightly decreases, the apple will fall. The spectators, who think that the string passes straight through the apple, are filled with wonder at the strange motion of the apple bobbing up and down.

140

CENTER OF GRAVITY

Many amusing mechanical tricks depend upon the principle that the center of gravity of a system of bodies always takes the lowest possible position; thus, a system will be stable if its center of gravity, when slightly disturbed, tends to fall to its original position.

To balance a cork upon the small end of a cane held vertically with that end upward. Put the prongs of two forks into opposite sides of the cork, letting the forks incline downward at angles of about 30 degrees with the vertical. Then the center of gravity of the cork and forks will be below the bottom of the cork, and thus there will be no danger of its falling off the end of the cane. The cane can be carried around held in a vertical position with the cork thus balanced on it.

A cork with two forks thus attached may be made to walk along a horizontal bar. Put two pegs of equal lengths into the bottom of the cork to act as legs, one being slightly in advance of the other. Then place the cork with its forks upon the horizontal bar, and set the forks into oscillation like a pendulum, the oscillations being parallel to the plane of the bar. The cork will then be alternately supported upon one of the two legs, and hence will advance or walk along the bar as long as the oscillations continue.

141

When the vertical line through the center of gravity lies without the base of support, the body will fall over, but when it lies within the base it will not fall. A toy horse standing with only his hind feet on the edge of a table will not fall if a curved wire attached to his breast runs backward and has a ball of sufficient weight at the free end. The horse may be made to rock to and fro without danger of falling, if the center of gravity of the horse and ball always rises when disturbed and if the vertical line through that center does not fall beyond the edge of the table.

142

INERTIA

Take several of the round wooden pieces which are used in playing checkers and put them in a vertical pile on a table. Then with a heavy knife blade strike the lowest block very quickly in a direction exactly parallel to the surface of the table. The lowest block will then move out under the impact of the blow but those above it will not be disturbed except that the whole pile will fall vertically to the table. This is an illustration of the doctrine of inertia, for there is no reason why the pile should move laterally unless it receives some impact from the blow; but this does not occur owing to the slight friction between the wooden pieces and to the suddenness with which the force is applied.



Fig. 56

The principle of inertia is utilized by the Japanese in a simple device (Fig. 57) for preventing the overthrow of

their pagodas by earthquakes. From the roof *A* of the pagoda there is suspended a heavy ball *B* by a wooden pendulum rod. When the earthquake comes the foundation of the pagoda is moved laterally to and fro and with it the lower part of the walls. The ball *B*,



Fig. 57

however, does not move until the motion can be communicated to it from the roof through the suspending rod. As this is a slow process the top *A* of the pagoda suffers only a slight lateral motion, and hence the structure is prevented from being overturned by the earthquake.

The seismograph used for recording vibrations due to earthquakes depends upon a similar principle. A heavy ball is so arranged, usually at the end of a horizontal pendulum, that it remains practically at rest while the ground moves laterally from the earthquake shock. Attached to the ball is a pointer touching lightly a sheet of paper on the recording apparatus which rests on the ground or floor. As this paper moves to and fro, the stationary pencil traces a curve which shows the intensity and duration of the earthquake shocks.

143

The cause of inertia may be imagined to be a change in size or shape of the atoms of the body due to action of the ether. Thus when a force puts a body in motion the atoms assume new shapes or sizes and thus store up energy. When the moving body meets resistance this energy is expended in overcoming that resistance, and the velocity of the body decreases. When a body comes to rest it

cannot move again under the action of a force until the atoms have assumed new forms and thus stored up the energy imparted by the force.

GRAVITATION

144

Gravitation is the great unsolved puzzle in the mechanics of the universe. The law of gravitation, namely, that any two atoms of matter attract each other with a force proportional to the product of their masses and inversely as the square of the distance between them, states merely observed facts and gives no clue as to the cause. The word attraction is perhaps an unfortunate one, for it implies that each body pulls upon the other. This might be true if each atom were joined to all other atoms by stretched elastic threads for the transmission of the force, but otherwise it is difficult to account for the force of pull. In fact, instances of pull are rare in mechanics; we say that the horse pulls the wagon, but in reality the horse pushes by his shoulders against the harness. The more rational explanation of gravity is that two bodies are pushed together by pressure exerted upon them from the space beyond their line of junction. To account for this push, LeSage supposed that multitudes of fine particles are moving in every direction through space. If there was only one body in the universe, these particles would impinge upon it from every direction and hence no motion of the body could occur. But for two bodies, it is plain that each will intercept particles that cannot fall upon the other, so that the bodies will be pushed together. While this accounts for the law of gravitation, it is of course no proof at all of the correctness of the theory, and there is no evidence at all of the fine moving particles.

Under the hypothesis of an ether which fills all space, the facts of gravitation require that bodies must be pushed together by the pressure of this ether. When two bodies are separated to a distance by applied forces, energy becomes stored in the ether; when the forces are removed this energy exerts pressures on each body which causes them to move toward each other. This general statement is about as far as we can go in explaining the cause of gravitation, but this rests upon the hypothesis of a universal ether, the existence of which has not been proved by any experimental facts.

145

Many absurd speculations regarding the cause of gravity have been made, and the following, from a pamphlet of 1893 called "Invisible and Visible," is one of the worst. "Gravitation is caused by the earth moving so fast that it draws everything to it, like a train of cars (when you stand close to the track) as it is passing."

Magnetic or electric action can be prevented from being propagated to a distance by screens of suitable material, but nothing has ever been discovered by which the action of gravitation can be screened off. The attraction of the earth acts with the same power upon a body, whether or not other bodies be interposed between it and the earth. Years ago it was recognized that the problem of flying would be solved if by any means a flying machine could be wholly or partially relieved from the attraction of the earth.

In 1847 Orrin Lindsay published at New Orleans a pamphlet entitled "Plan of Aerial Navigation, with a Narrative of his Explorations in the higher Regions of the Atmosphere and his wonderful Voyage around the Moon."

His "plan" consisted in annulling the force of gravity. Well-prepared steel, after being superficially coated, amalgamated with quicksilver, and then strongly magnetized, proved to be an impervious screen to gravitation. A hollow box made of these metal plates, rose from the earth; to cause it to descend, a hole was opened in the bottom; to cause it to move laterally, a hole was opened in the side. It is unnecessary to explain here his voyage to the moon.

About 1900 there was published a novel by Simon Newcomb called "His Wisdom the Defender," in which flights by a huge machine were made by its property of annulling the force of gravity. The inventor and owner made aerial voyages over the earth, and compelled the nations to disband their armies under the threat of dropping bombs which would blow their cities into nothingness. Thus this inventor, who was called "His Wisdom," inaugurated upon the earth a reign of universal peace.

146

One of the most interesting papers on the ether of space is that of DeVolson Wood in the London Philosophical Magazine of November, 1885. It is based on the known facts: (1) that the ether transmits light at a velocity of 186 300 miles per second; (2) that the ether transmits 133 foot-pounds of energy per second from the sun to each square foot of the earth's surface. His discussion leads to the conclusions (1) that the mass of a cubic foot of the ether at the earth's surface is 2×10^{-24} pounds, (2) that the ether has an elasticity such that it exerts a pressure of 4×10^{-8} pounds on each square foot of the earth's surface, (3) that the ether has the enormous specific heat of 4 600 000 000 000, so that to raise one pound of it 1° F. would require as much

heat as it would to raise 2 300 000 000 tons of water the same amount. This medium, says Wood, will be everywhere practically non-resisting and sensibly uniform in temperature, density, and elasticity. In one pound of it there is 10^{10} times the kinetic energy of a pound of gas.

147

THE DIAPHOTE HOAX

From a Pennsylvania daily newspaper of Feb. 10, 1880.

A special meeting of the Monacacy Scientific Club was held on Saturday evening to listen to a paper by Dr. H. E. Licks on the diaphote, an instrument invented by him after nearly three years of study, and now so nearly perfected that he feels warranted in bringing some few of the results thus far attained to the notice of the public. There were present, besides many scientists of Eastern Pennsylvania, Prof. M. E. Kannick of the polytechnic school at Pittsburg, and Col. A. D. A. Biatic of the Brazilian corps of engineers, who is now in this country making extensive purchases of iron and steel. The meeting was called to order by the president, Prof. L. M. Niscate, who in introducing Dr. Licks made a few remarks, saying that he had had an opportunity to witness a few experiments with the diaphote, and he felt convinced that it would ultimately rank with the telephone, the phonograph, and the electric light as one of the most remarkable triumphs of science in the nineteenth century.

Dr. Licks prefaced his paper by saying that the idea of the invention was first suggested to his mind about three years before by reading accounts of some of the early experiments of Bell's telephone, and that a little later when Edison brought out the carbon instrument, his studies had become so far advanced as to assure him of its theoretic

possibility. By the telephone the sound of the human voice may be heard hundreds of miles away. Why, then, cannot light be transmitted in a similar manner, so that by the use of a connecting wire one may distinctly see the image of the object far removed? This, said Dr. Licks, was the form in which the inquiry first suggested itself to him nearly three years ago, and he felt gratified to be able to exhibit to the club this evening an instrument called the diaphote in which the practical realization of the idea had been in a great measure satisfactorily obtained. The word diaphote, from the Greek *dia* signifying through, and *photos*, signifying light, had been selected as its name, implying that the light travelled through or in the wire. Although popularly this might be imagined to be the case, it was really no more so than with sound in the telephone. There the sound waves strike a diaphragm that is set into vibration, and generates induced electricity in the wire, this causing corresponding vibrations in another distant diaphragm which reproduces similar sounds. In the diaphote, likewise, the waves of light from an object strike a peculiarly constructed mirror or speculum which is joined by a wire with another similar speculum; the image of an object in the first modifies the electric current in the wire and passing quickly onward to the receiving instrument produces there a secondary image. The intermediate wire, as in the telephone, may be hundreds of miles in length, yet such is the delicacy of the diaphotic plates that the transmitted image of a simple object is almost as distinct as the original, and Dr. Licks feels confident that after the removal of a few obstacles, of a mechanical nature only, the most complex forms will be reproduced with the strictest fidelity as to outline and color.

The diaphote consists of four essential parts, the receiving mirror, the transmitting wires, a common galvanic battery, and the reproducing speculum. Dr. Licks gave a detailed account of the experiments to determine the composition of the mirror and speculum. For the former he had finally selected an amalgam of selenium and iodide of silver, and for the latter an amalgam of selenium and chromium. The peculiar sensitiveness of iodide of silver and chromium to light has long been known and their practical use in photography suggested their application in the diaphote. It was found, however, after many experiments, that their action must be so modified that each ray of light should influence the electric current proportionally to its position in the solar spectrum, and the element selenium was selected as best adapted to this purpose. At first a small mirror was employed with only a single wire, but the images in the speculum were confused and indistinct so that it became necessary to make the mirror of pieces each about one-third of a square inch in area and each having a wire attached. In the diaphote exhibited by Dr. Licks to the club, the mirror was six by four inches in size, and there were 72 wires which were gathered together into one about a foot back of the frame, the whole being wrapped with insulating covering; and in reaching the receiving speculum each little wire was connected to a division similarly placed as in the mirror. From a galvanic battery wires ran to each diaphotic plate and thus a circuit was formed which could be opened or closed at pleasure. Dr. Licks explained how the light caused momentary chemical changes in the mirror which modify the electric current and cause similar changes in the remote speculum, this causing a similar image which may be readily seen or be thrown upon a screen by a second

camera. He explained how the proportions of selenium should be scientifically adjusted to the resistance of the electric current so as to avoid any blending of the reproduced images. This, he said, had been the problem which had caused him the most trouble and which at one time had seemed almost insurmountable.

At the close of the paper an illustration of the powers of the instrument was given. The mirror of the diaphote, in charge of a committee of three, was taken to a room in the lower part of the building, and the connecting wires were laid through the halls and stairways to the speculum on the lecturer's platform. Before the mirror, the committee held in succession various objects, illuminating each by the light of a burning magnesium tape, since the rays from gas are deficient in actinic power; simultaneously on the speculum appeared the reproduced images, which for exhibition to the audience were thrown on a screen considerably magnified. An apple, a penknife, and a trade dollar were the first objects shown; in the latter the outlines of the goddess of liberty were recognized and the date 1878 was plainly legible. A watch was held for five minutes before the mirror and the audience could plainly perceive the motion of the minute hand, but the motion of the second hand was not satisfactorily seen, although Prof. Kannick by looking into the speculum said it was there quite perceptible. An ink bottle, a flower, and a part of a theater handbill were also shown, and when the head of a little kitten appeared on the screen the club expressed its satisfaction by hearty applause.

After the close of the experiments the scientists congratulated Dr. Licks on his invention, and the president made a few remarks on the probable scientific and industrial

applications of the diaphote in the future. With telephone and diaphote it may yet be possible for friends far apart to hear and see each other at the same time, to talk, as it were, face to face. In connection with the interlocking switch system it may be used to enable the central office to see many miles of track at one time, thus lessening the liability to accident. In connection with photolithography it could be so employed that the great English papers could be printed in New York a few hours after their appearance in London. Our reporter also learned that Dr. Licks will lecture on the diaphote next week before the American Society of Arts, and that he will make definite arrangements for the manufacture of the instrument as soon as the seven patents for which he has applied are formally issued.

Within a week after the publication of the above article, it was copied in whole or in part by numerous papers throughout the United States, many commenting editorially on the great possibilities of the marvellous diaphote. Some papers said that sunlight would be transmitted by it from the sunny side of the earth to light the side which was in darkness. The *New York Times* said "the imagination almost fails before the possibilities of what the diaphote may yet accomplish." The only paper which recognized the article as a fake seems to have been the *New York World*, which said, "the hoax is a clever one and is interesting also as depending for its success upon the opposite of the mistake which was at the bottom of Locke's famous 'Moon Hoax'; it is the misuse of the word mirror in connection with the new 'invention' which has made the miracle of it so acceptable to the public." Within a month after the publication of the diaphote hoax, items appeared in the papers announcing the invention in Pittsburg of an instrument called the

“telephole” by which two persons at a distance could see each other as they talked over the telephone, and by which any written or printed document could be transmitted instantaneously to any distance. The inventors of this instrument, it was stated, had labored many years in making experiments and now success had been attained. While Dr. Licks used 72 wires, the Pittsburg inventors used but one, and their applications for patents were soon to be granted.

News of the diaphote soon spread to Europe, and in due time there came back to us stories of wonderful inventions there made. For instance in 1889, the news came that a young German, named Korzel, exhibited an instrument by which a person in one city could read a newspaper held before a receiving plate in another distant city. The secret of this marvellous instrument, it was said, lay in the sensitiveness of selenium to the effects of light, its electric conductivity changing with the color and intensity of the light which impinged upon the plate. Very curiously all the inventors of such instruments have used selenium since its properties were first utilized in the diaphote by Dr. Licks. Almost every year similar stories have appeared, the most recent being one which was published in the *New York Times* of May 29, 1914, in the form of a cable dispatch from London. This article states that on the previous day, Dr. A. M. Low, a well-known scientific investigator, lectured before the Institute of Automobile Engineers on “Seeing by Wire.” For five years his experiments had been carried on and now he had attained such success that pictures were reproduced at a distance of four miles. His instrument “has a receiving screen consisting of a large number of cells of selenium, over which a ruler is moved rapidly by a small

motor worked with a current of high frequency and about 50 000 volts pressure. The receiver at the other end is made up of a series of telephone slabs of steel, through which the light passes." Perhaps Dr. Low is on the right track, and if his apparatus becomes a verity, then he should give proper credit to Dr. H. E. Licks by calling it the diaphote.

148

THE ONE-HOSS SHAY

The secret of successful engineering construction is to make each part of a structure just as strong as the other parts, so that there can be no weak spot where failure may occur. Oliver Wendell Holmes wrote many years ago a delightful poem on this principle. It begins:

Have you heard of the wonderful one-hoss shay
That was built in such a logical way
It ran a hundred years to a day?

The "shay" was supposed to have been built by a Deacon in Massachusetts who was resolved that it should be properly constructed.

But the Deacon swore (as deacons do)
It should be so built that it couldn't break down,
"Fur," said the Deacon, "'tis mighty plain
That the weakest spot must stan' the strain,
And the way to fix it, as I maintain, is only jest
To make that place as strong as the rest."

The wheels were just as strong as the thills,
And the floor was just as strong as the sills,
And the panels just as strong as the floor,
And the whipple-tree neither less or more.
And the back cross bar as strong as the fore
And spring and axle and hub encore.

The chaise was designed to run exactly a hundred years, and so it did. When that time arrived a parson was riding in it and the catastrophe came.

All at once the horse stood still,
Close by the meetin'-house on the hill,
First a shiver, and then a thrill,
Then something decidedly like a spill,
And the parson was sitting on a rock,
At half-past nine by the meetin'-house clock,

You see of course, if you're not a dunce
How it went to pieces all at once,
All at once and nothing first,
Just like bubbles when they burst.

End of the wonderful one-hoss shay,
Logic is logic! That's all I say.

CHAPTER IX

APPENDIX

149



ONCE upon a time a man, after much labor, raised a number of two digits to the 31st power this containing 35 digits. Stating this fact to a lightning calculator, he was about to give the long number, when the calculator said that this was unnecessary and that the root was 13. How did he know this? Simply from having committed to memory a table of two-place logarithms and by making a rapid computation from them. Since the given power has 35 digits its logarithm lies between 34.00 and 35.00. Dividing these by 31 gives 1.09 and 1.13 as the logarithms of numbers between which the root must lie. Then, remembering that the logarithms of 12, 13, and 14 are 1.08, 1.11, and 1.15 the computer instantly saw that the required number must be 13. Hence, the man who computed that 34 059 943 367 449 284-484 947 168 626 829 637 was the 31st power of 13 had his labor for his pains, for there was no opportunity to give a single figure of it to the lightning calculator. In fact 13 is the only number of two digits whose 31st power has 35 digits.

150

MERSENNE'S NUMBERS

In 1644 Pere Mersenne made certain statements regarding numbers of the form $2^p - 1$ where p is a prime. These statements seem to be that the only values of p , not greater

than 257, which make $2^p - 1$ a prime, are 1, 2, 3, 5, 7, 13, 17, 31, 61, 127, 257 and that it is composite for all other values of p . Thus, $2^{11} - 1 = 2047 = 23 \times 89$, and $2^{23} - 1 = 2\,388\,607 = 47 \times 178\,481$. How he arrived at these conclusions is a mystery, but it is supposed to have been through correspondence with the great mathematician Fermat.

There are 56 primes not greater than 257. Mersenne's statement has been verified for 38 of these, namely, for 10 of the twelve values of $2^p - 1$ which he stated to be prime, and for 28 of the 44 values which he stated to be composite. Although much acute thought has been spent upon them by great mathematicians like Euler and Gauss, yet 18 values of $2^p - 1$ are yet unverified, namely, for $p = 89, 101, 103, 107, 109, 127, 137, 139, 149, 157, 167, 173, 193, 199, 227, 229, 241, 257$.

Fermat, in 1679, gave a rule for determining factors of the number $2^p - 1$. He said, in effect, that if 2 or 8 or 32 be subtracted from a perfect square, the remainder n will generally divide $2^p - 1$ when n is a prime and $n - 1$ is a multiple of p . Thus, from 25 take 2, the remainder 23 divides $2^{11} - 1$ since 23 is prime and $23 - 1$ is a multiple of 11. From 49 take 2, the remainder 47 divides $2^{23} - 1$ since 47 is prime and $47 - 1$ is a multiple of 23. From 225 take 2, the remainder 223 divides $2^{37} - 1$ since 223 is prime and $223 - 1$ is a multiple of 37. These three illustrations of the process are given by Fermat.

The reason for this rule is unknown to me, but following the same line of procedure I take 2 from 169 and 167 is known to be a factor of $2^{83} - 1$; also taking 2 from 361 the remainder 359 is a factor of $2^{179} - 1$; also taking 2 from 441 the remainder 439 is a factor of $2^{73} - 1$. Further, taking 32 from 121 the remainder 89 is known to be a factor of

$2^{11} - 1$. But this method breaks down when 32 is taken from 841; here the remainder is 809 and is prime and 808 is a multiple of 101, hence it might be expected that 809 is a factor of $2^{101} - 1$, but on trial this is found not to be the case, the division yielding a remainder of 491. Fermat's method gives a factor for some values of $2^p - 1$ but it fails in others.

The factorization of large numbers is a very difficult subject. Some values of $2^p - 1$ have large factors; see Bulletin of American Mathematical Society for December, 1903, where Cole shows that 193 707 721 and 761 838 257 287 are the factors of $2^{67} - 1$. For a very interesting history of the work done on Mersenne's numbers see Ball's Mathematical Recreations and Essays.

151

In 1850 the Rev. T. P. Kirkman proposed the following problem in the *Lady and Gentlemen's Diary*, an annual published in England: A schoolmistress takes her fifteen girls out for a walk every day in the week; they are arranged in five rows, each row containing three girls; how can they be arranged for a full week so that no girl will walk with any of her schoolmates more than once?

This is generally known as Kirkman's School Girls Problem, and it has been discussed by many mathematicians. The following is Kirkman's solution:

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
$a_1a_2a_3$	$a_1b_1c_1$	$a_1d_1e_1$	$a_1b_2d_2$	$a_1c_2e_2$	$a_1b_3e_3$	$a_1c_3d_3$
$b_1b_2b_3$	$a_2b_2c_2$	$a_2d_2e_2$	$a_2b_3d_3$	$a_2c_3e_3$	$a_2b_1e_1$	$a_2c_1d_1$
$c_1c_2c_3$	$a_3d_3e_3$	$a_3b_3c_3$	$a_3c_1e_1$	$a_3b_1d_1$	$a_3c_2d_3$	$a_3b_2e_2$
$d_1d_2d_3$	$b_3d_1e_3$	$d_3b_1c_2$	$b_1c_3e_2$	$c_1b_3d_2$	$b_2c_3d_1$	$c_2b_3e_1$
$e_1e_2e_3$	$c_3d_2e_1$	$e_3b_2c_1$	$d_1c_2e_3$	$e_1b_2d_3$	$e_2c_1d_3$	$d_2b_1e_2$

He also showed that there are four other solutions, so that the schoolmistress might take out her fifteen young ladies every day for five weeks without any girl walking with any of her mates more than once in a triplet. Ball's *Mathematical Recreations and Essays* devotes 31 pages to this problem but gives no clear solution of it. In 1862, Sylvester claimed that the girls could walk every day for thirteen weeks under the final condition of the problem.

152

DETERMINANTS

A determinant is an abridged notation for certain algebraic operations to be performed. The theory arose from the formulas required for solving simultaneous equations of the first degree. Thus, when there are two equations containing two unknown quantities,

$$a_1x + b_1y = c_1, \quad a_2x + b_2y = c_2,$$

the solution gives for the values of x and y ,

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

which may be written in abridged notation as follows:

$$x = \left| \begin{array}{cc} c_1 & b_1 \\ c_2 & b_2 \end{array} \right| \div \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right| \quad y = \left| \begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array} \right| \div \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right|.$$

The quantities enclosed within the parallel lines are the simplest possible determinants. $\left| \begin{array}{cc} a_1 & c_1 \\ a_2 & c_2 \end{array} \right|$ means that the

quantities in the diagonal inclined downward to the right are to be multiplied together giving a_1c_2 , and that from this is to be subtracted the product of the quantities in the other diagonal or a_2c_1 ; thus, $a_1c_2 - a_2c_1$ is the value of this determinant.

Again let there be three equations having three unknown quantities, namely,

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3.$$

Here the value of x in the determinant notation is

$$x = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in which the first determinant has the value

$$b_1 \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} - b_2 \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} + b_3 \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

and this becomes by expanding each minor determinant,

$$b_1 (c_2d_3 - c_3d_2) - b_2 (c_1d_3 - c_3d_1) + b_3 (c_1d_2 - c_2d_1).$$

Similarly, the second determinant in the value of x is

$$a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1).$$

Here the signs of the second, fourth, etc., minor determinants must be minus, while the others are plus.

Determinants have many interesting properties, only one of which can here be mentioned. If any two adjacent rows or adjacent columns be interchanged, the sign of the determinant changes. Thus the three determinants:

$$\begin{vmatrix} +3 & -1 & 0 \\ +4 & +2 & +6 \\ +1 & -3 & +2 \end{vmatrix} \quad \begin{vmatrix} -1 & +3 & 0 \\ +2 & +4 & +6 \\ -3 & +1 & +2 \end{vmatrix} \quad \begin{vmatrix} +3 & +1 & 0 \\ +1 & -3 & +2 \\ +4 & +2 & +6 \end{vmatrix}$$

have the values +68, -68, and -68. When one or more of the quantities in a column is 0, it is convenient to interchange so as to make this the first column, and no change of

sign will result if an even number of columns is passed over, thus:

$$\begin{vmatrix} 0 & +3 & -1 \\ +6 & +4 & +2 \\ +2 & +1 & -3 \end{vmatrix} = 0 - 6(-9 + 1) + 2(6 + 4) = +68.$$

153

The notation " $a \equiv b \pmod{c}$ " means that $a - b$ is exactly divisible by c . Thus, $2^{n-1} \equiv 1 \pmod{n}$ is a true statement whenever n is a prime number greater than 2; for example, let $n = 7$, then $2^6 - 1 = 63$ which is exactly divisible by 7. The expression $a \equiv b \pmod{c}$ is called a congruence and the abbreviation "mod" means modulus, it being said that a and b are congruent for the modulus c . This notation is mentioned here because many students who see it in mathematical literature know not its meaning.

While it is true that $2^{n-1} \equiv 1 \pmod{n}$ when n is an odd prime, it is not true that n must be a prime to satisfy the congruence. In fact $2^{340} \equiv 1 \pmod{341}$, although 341 is not a prime number. These facts and numerous other interesting ones are proved in that branch of Higher Algebra known as the theory of numbers.

154

To find the date of Easter Sunday for any year in the twentieth century: Divide the given year by 19 and call the remainder a , divide it by 4 and call the remainder b , divide it by 7 and call the remainder c ; divide $19a + 24$ by 30 and call the remainder d ; divide $2b + 4c + 6d + 5$ by 7 and call the remainder e . Then Easter is March $(22 + d + e)$ or April $(d + e - 9)$. In 1954 and 1981, however, Easter comes one week later than the date given

by this rule. Example: To find Easter for 1916: here $a = 16$, $b = 0$, $c = 5$, $d = 28$, $e = 4$; then the date of Easter Sunday is April 23. The earliest date on which Easter can fall is March 22 and the latest date is April 25.

155

HOW TO DRAW A STRAIGHT LINE

The pantograph used for copying maps on an enlarged or reduced scale is the simplest apparatus in which one point always moves parallel to another. Fig. 58 shows its simplest form, $ABCD$ being a parallelogram with joints at the corners and fastened to the paper at A . The straight rod EFP is fixed to AD at E and can slide longitudinally along a peg F which is fixed to BC at F . When the point p is

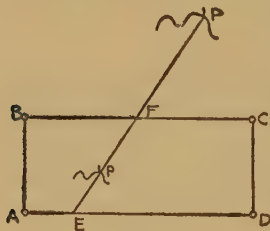


Fig. 58

moved along a line of the map a pencil at the point P traces a line everywhere parallel to p but on an enlarged scale. When the line followed by p is straight the point P traces a straight parallel line.

Fig. 59 shows the principle of an apparatus by which a point E can be made to move in a straight line to A . The bar ED slides at D in the fixed groove DB along a straight line which always passes through the fixed point A , and the length AC equals EC or CD . As the point C is made to rotate about A in a circle Cc the end D slides along the

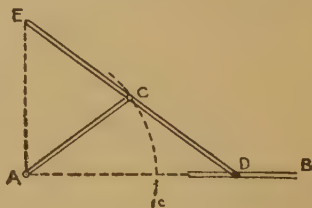


Fig. 59

groove and thus the point E moves in the straight line EA .

Euclid's postulate that a straight line can be drawn gives no information as to how this may be done. It is usually supposed to be drawn by means of a ruler, but then the question arises as to how the ruler has been made straight.

In 1864 this problem was solved by Peancellier by means of a linkwork in which a point C moves in a circle Cc about a fixed point B while another point E traces a straight line ee . Here the parallelogram $CDEF$ is jointed at the

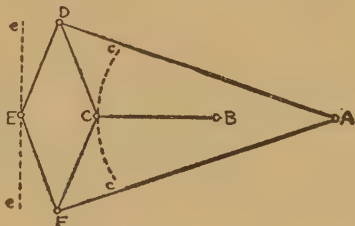


Fig. 60.

corners, and the radius arm BC is one-half of the distance CA , and A is also a fixed point which is connected with the parallelogram by the rods AD and AF . Thus the circular motion of C causes the right-line motion of E . Since there can be no doubt about the circle, this apparatus solves the problem as to how a straight line can be drawn.

156

MISCELLANEOUS FALLACIES

Plot the following points on the rectangular coördinate system and join the successive points by a curve; the first number in each parenthesis being the value of x and the second the value of y : (7.0, 4.5), (6.0, 4.7), (4.0, 4.5), (2.0, 3.6), (1.0, 2.9), (0.6, 2.7), (0.3, 2.0), (0.6, 1.0), (1.0, 0.7), (2.0, 0.6), (3.0, 0.8), (3.5, 1.8), (4.0, 0.7), (4.5, 0.6), (5.0, 0.6), (4.8, 1.6), (5.0, 1.8), (5.2, 2.0), (5.5, 2.5), (7.0, 2.6), (7.5, 2.6), (7.7, 2.9), (7.3, 2.9), (7.0, 3.5), (7.0, 4.5). When

the reader has drawn this interesting curve, let it receive careful study.

157

The camber of a bridge is a slight upward curve given to the floor so that the structure may have the appearance of strength and stiffness. In the days of early American engineering an excessive camber was given to a certain railroad bridge; it is said that the superintendent received reports that the piers were sinking so as to leave the middle of the structure higher up. Whether this story be true or not, it is certain that the following was clipped from a newspaper printed in 1879 in the Pennsylvania German region along the Delaware River:

"It's warm, Louis, ain't it," said Tod Hartzell to Louis Rapp, as they met on the Delaware bridge yesterday. "Oh, vell, it is," said Louis, "how much you vay now Tod?" "Only 288 pounds," said Tod. "I can beat that," said Louis, "for I vay 294 pounds." The bystanders, by mental arithmetic, added the weights of the men, and then hurried from the bridge which cracked and groaned under the enormous load, while Louis and Tod gracefully moved from the center of the arch which then sprung back to its original position.

158

The following fallacy is taken from an old newspaper where it is dignified by the title "A Scientific Lecture on Glass."

A neat, simple, and quick way of punching a hole through a glass plate is, I venture to say, ladies and gentlemen, unknown to most of you. Nothing can be easier than to punch such a hole and at the same time to cause the utter destruction of the glass, but it is not of this that I am to speak, but rather of a simple scientific operation by which anyone may punch or drill a small hole through a plate of window glass, without injuring it or cracking it in the slightest degree. The tools necessary for this purpose are two sets of punches, an old file, and a heavy hammer, which every mechanic possesses. Armed with these, each of you may become skilled in this most interesting and useful accomplishment.

The thicker the pane of glass on which you are to operate, the easier is the process. Having selected it, you choose a place not too near the edge, and with the end of the old file scratch two marks upon it crossing each other like the letter *x*. Then turn the plate over and precisely opposite scratch a similar cross. Next select two set punches of the same size and fasten one of them securely in a vise. Let an assistant hold the plate in a horizontal position with the lower cross resting exactly on the fastened vertical punch, while you with the left hand hold the other punch on the upper cross, and with your right hand grasp the heavy hammer. You then elevate the hammer, but when you strike be careful to give only a moderate blow, for a violent one might cause the destruction of the glass. The effect of the blow, if it be scientifically directed, will be to cause a very slight indentation in the glass. Then let your assistant turn the plate over and again balance it upon the fastened punch, while you with the hammer repeat the careful blow. The indentation will now be more marked than before, and by repeating the process half a dozen times a hole will be made entirely through the glass plate which will be as finely cut as if produced by a swiftly moving rifle ball, while no crack will appear.

When I was captured by the Waldamites this accomplishment proved of the greatest benefit to me, and in fact enabled me ultimately to escape. These singular people, although excellent glass workers, knew no way of cutting it, and when they saw how readily I punched such holes, they not only made obeisance before me, but what was better, they gave me boiled rice and roc's eggs to eat, which were very acceptable, as for six weeks I had eaten only roots with now and then an herb.

QUOTATIONS

159

Ἐν τοῖς ὀρθογώνιοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετραγώνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

EUCLID, Elements, Book I, 300 B. C.

160

Πᾶς κύκλος ἴσος ἐστὶ τριγώνῳ ὀρθογωνίῳ οὗ ἡ μὲν ἐκ κέντρου ἰση μὲν τῶν περι τὴν δρθήν, ἡ δὲ περίητρος τῇ βασει.

ARCHIMEDES, Measurement of the Circle, 220 B. C.

Translation by Haurer, 1798: Jeder Kreis ist einem rechtwinklichen Dreyek gleich, dessen eine Seite um den

rechten Winkle dem Halbmesser und die andere dem Umfang des Kreises gleich ist.

161

Sic incertum, ut, stellarum numerus par an impar sit

CICERO, *Academia*, about 50 B. C.

162

This is the third time; I hope, good luck lies in odd numbers. . . . They say, there is divinity in odd numbers, either in nativity, chance, or death.

SHAKESPEARE, *Merry Wives of Windsor*, 1599.

163

And now we might add something concerning a certain most subtile Spirit which pervades and lies hid in all gross bodies, by the force and action of which Spirit the particles of bodies mutually attract one another at near distances and cohere, if contiguous; and electric bodies operate to greater distances, as well expelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this Spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain to the muscles. But these are things that cannot be explained in a few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic Spirit operates.

NEWTON, *Principia*, Book III. 1687: American edition, 1848.

164

Vedete hora quanto mirabilmenti si accordano còl sistema Copernicano queste tre prime corde, che da principio parevan si dissonanti. Di qui potra instantly . . . vedere con quanto probabilità si possa concludere, che non la terra, ma il Sole sia nel centra delle conversioni de i pianetti. E poichè la terra vien collocata tra i corpi mondani, che indubitatamente si muovono intorno al Sole, cioè sopra Mercurio, e Venere, e sotto a Saturno, Giove, e Marti, comme parimente non sara probabilissimo, e forse necessario concedera, che essa ancora gli vadia interno?

GALILEI, Third Dialogue, 1630.

165

In philosophia experimentali, propositiones ex phaenomenis per inductionem collectae, non obstantibus contrariis hypothesibus, pro veris aut accurate aut quamproxime haberi debent, donec alia occurrerint phaenomena per quas aut accuratiores reddantur aut exceptionibus obnoxiae.

NEWTON, Principia, Book III, 1687.

166

It is said that the Egyptians, Persians, and Lacedaemonians seldom elected any new kings but such as had some knowledge in the mathematics; imagining those who had not, to be men of imperfect judgements, and unfit to rule and govern.

Though Plato's censure that those who did not understand the 117th proposition of the 13th book of Euclid's Elements ought not to be ranked among rational creatures, was unreasonable and unjust, yet to give a man character

of universal learning, who is destitute of a competent knowledge in the mathematics, is no less so.

FRANKLIN, Usefulness of Mathematics, 1735.

167

Dieu parle, et le chaos se dissipe à sa voix:
Vers un centre commun tout gavite à la fois.
Ce ressort si puissant, l'ame de la nature,
Était enseveli dans une nuit obscure:
Le compas de Newton, mesurant l'univers,
Leve enfin ce grand voile, et les cieus s'ouvrent.

VOLTAIRE, Letter to Madame Châtelet, 1735.

168

On s'imagine que toutes ces étoiles, prises ensemble, ne constituent qu'une très-petite partie de l'univers tout entier. À l'égard duquel ces terribles distances ne sont par plus grandes qu'un grain de sable par rapport à la terre. Toute cette immense est l'ouvrage du Tout-Puissant, qui gouverne également les plus grandes corps, comme les plus petits, et qui dirige le succès des armes, auquel nous sommes intéressés.

EULER, Letters to a German Princess, 1760.

169

Geheimnissvoll am lichten Tag
Lässt sich Natur des Schleiers nicht berauben,
Und was sie deinem Geist nicht offenbaren mag,
Das zwingst du ihr nicht ab mit Hebeln und mit Schrauben.

GOETHE, Faust, Part I, 1790.

170

Der Gebrauch einer unendlichen Grösse als eine Vollendetes ist in der Mathematik niemals erlaubt. Das Unendliche ist nur eine Façon de parler, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kom-

men als man will, während anderen ohne Einschränkung zu wachsen verstattet ist.

GAUSS, Letter to Schumacher, 1831.

171

Then Rory, the rogue, stole his arm round her neck,
 So soft and so white, without freckle or speck;
 And he look'd in her eyes, that were beaming with light,
 And he kissed her sweet lips — don't you think he was right?
 "Now, Rory leave off, sir, you'll hug me no more,
 That's eight times today that you've kissed me before."
 "Then here goes another," says he, "to make sure,
 For there's luck in odd numbers," says Rory O'More.

LOVER, Rory O'More, 1839.

172

There are terms which cannot be defined, such as number and quantity. Any attempt at a definition would only throw a difficulty in the student's way, which is already done in geometry by the attempts at an explanation of the terms point, straight line, and others, which are to be found in treatises on that subject. A point is defined to be that "which has no parts and which has no magnitude"; a straight line is that which "lies evenly between its extreme points." . . . In this case the explanation is a great deal harder than the term to be explained, which must always happen whenever we are guilty of the absurdity of attempting to make the simplest ideas yet more simple.

DE MORGAN, On the Study of Mathematics, 1831.

173

All the mathematical sciences are founded on relations between physical laws and laws of numbers. so that the aim

of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers.

MAXWELL, Faraday's Lines of Force, 1853.

174

Pour les astres en général at pour les grand Comètes en particulier, trois mille ans ne sont pas grand'chose: dans le calendrier de l'éternité c'est moins qu' une seconde. Mais pour l'homme vous savez comme moi, mathématicien lecteur, que trois mille ans c'est boucoup, beaucoup!

FLAMMARION, Recits de l'infini, 1892.

175

There still remain three studies suitable for freemen. Calculation in arithmetic is one of them; the measurement of length, surface, and depth is the second; and the third has to do with the revolutions of the stars in reference to one another.

PLATO, Republic, 350 B. C., Jowett's Translation, 1894.

176

The heavens themselves, the planets, and this centre,
Observe degree, priority, and place,
Insisture, course, proportion, season, form,
Office, and custom, in all line of order;
And therefore is the glorious planet, Sol,
In noble eminence enthron'd and spher'd
Amidst the others; whose med'cinable eye
Corrects the ill aspects of planets evil,
And posts, like the commandment of a king,
Sans check, to good and bad: but when the planets
In evil mixture to disorder wander,
What plagues and what portents? what mutiny?
What raging of the sea? frights, changes, horrors,
Divert and crack, rend and deracinate
The unity and married calm of states
Quite from their fixture.

SHAKESPEARE, Troilus and Cressida, Act I, Scene 3, 1602.

177

Lassune' ke' nipune' ani tis de machir mirive' iche manir
se' de évenir toné chi amiché ze forime' to viche tarviné.

FLOURNOY, Des Indes a la planete Mars, 1900.

178

The following is one of the many stories told of "old Donald McFarlane," the faithful assistant of Sir William Thomson: The father of a new student when bringing him to the university, after calling to see the Professor (Thomson) drew his assistant to one side and besought him to tell him what his son must do that he might stand well with the Professor. "You want your son to stand weel with the Profeesor?" asked McFarlane. "Yes." "Weel, then he must just have a guid bellyful o' mathe-matics!"

S. P. THOMPSON, Life of Lord Kelvin, 1910.

179

Todhunter was not a mere mathematical specialist. He was an excellent linguist; besides being a sound Latin and Greek scholar, he was familiar with French, German, Spanish, Italian, and also Russian, Hebrew, and Sanskrit.

MACFARLANE, Ten British Mathematicians
of the Nineteenth Century, 1916.

180

The Appendix is the Soul of a Book.

Old Proverb, n. d.

